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OPTIMAL AERODYNAMIC AND THRUST MAGNITUDE CONTROL OF MANEUVERING ROCKETS

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Michigan University

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Optimal Aerodynamic and Thrust Magnitude Control of Maneuvering Rockets

Final Report

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13. ABSTRACT

This report presents the results of a study to evaluate thrust magnitude control in Air Force missions. In this study modern control theory was applied to determine the time history of the thrust profile to achieve optimal maneuvers of aerospace lifting vehicles. Following formulation of a mathematical model with a very general propulsion system, optimal control laws for the thrust direction and magnitude were obtained. It was shown that coast, boost, and sustaining thrust could all be optimal. For the particular case of sustaining flight it was shown that the position vector, the velocity vector, and the mass of the rocket vehicle must satisfy a certain explicitly obtained relationship, and that the variable thrust magnitude depends strongly on the aerodynamic characteristics of the vehicle and the optimal trajectory flown. Particular attention was given to the solution for minimum time lateral turns in a horizontal plane for a lifting missile. A qualitative analysis was then carried out for some typical missions. The numerical solutions indicate that the precentage performance gain with throttling increases with the difficulty of the mission (especially position target missions.)

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NOMENCLATURE

a, a_1, a_2, a_4, b	=	constants
A	=	$a_1 \cos\beta + a_2 \sin\beta$
$\vec{A}$	=	atmospheric force
c	=	exhaust velocity
C	=	constant of integration
C_D	=	drag coefficient
C_{D_o}	=	zero-lift drag coefficient
C_L	=	lift coefficient
C_{L_α}	=	slope of the lift curve
D	=	drag
E^*	=	maximum lift-to-drag ratio
$g, \vec{g}$	=	acceleration of the gravity (vector)
G	=	term independent of β and T (Eq. II.13)
H	=	hamiltonian
J	=	performance index
k	=	induced-drag coefficient
K	=	switching function
L	=	lift
m	=	mass of the rocket
M	=	mach number
p_i	=	adjoint variables
$\vec{p}_r, \vec{p}_v, p_m$	=	adjoint vectors (or variable)

P	=	power of the propulsion system
$\vec{r}$	=	position vector of the point mass
R	=	reference length
s	=	arc length
S	=	reference area
t	=	non-dimensional time
t'	=	real time
$\vec{T}$, T	=	thrust vector, magnitude
u	=	dimensionaless velocity
V	=	velocity
w	=	dimensionless weight
W	=	weight of the rocket
x	=	dimensionless longitudinal range
X	=	longitudinal range
y	=	dimensionless lateral range
Y	=	lateral range
α	=	angle of attack
β	=	heading angle
λ	=	lift control
λ_M	=	maximum lift control
Λ	=	$\sqrt{\lambda^2 u^4 - w^2}$
η	=	aerodynamic parameter
ρ	=	air mass density
ρ_o	=	air mass density at reference level
σ	=	bank angle

τ	=	dimensionless thrust
τ_M	=	maximum dimensionless thrust
μ	=	gravitational constant

I. INTRODUCTION

This report presents the results of a study to evaluate thrust magnitude control (TMC) in Air Force missions. In this study, modern control theory was applied to determine the ime history of the thrust profile to achieve optimal maneuvers of aerospace lifting vehicles.

This report consists of five parts. After an introduction, we give a general discussion of optimal thrust magnitude control in Chapter II. A mathematical model for a very general propulsion system, the system (S), was formulated. From this system we deduce the most commonly used system, the system (S_1), where the exhaust velocity of the gas ejected from the engine is constant. Optimal control laws for the thrust direction, and the thrust magnitude are obtained. It is shown that coast, boost, and sustaining thrust all may be optimal. In the singular case of sustaining flight it is shown that:

1. Along the sustaining flight path, the position vector $\vec{r}$, the velocity vector $\vec{V}$ and the mass m of the rocket vehicle must satisfy a certain relation obtained explicitly.

2. The variable thrust magnitude depends strongly on the aerodynamic characteristics of the vehicle, and the optimal trajectory flown.

The results in this chapter were obtained for a Newtonian gravitational force field. In the remaining part of the report we consider in detail the particular case of turning flight in a horizontal plane, in a uniform gravitational field.

In Chapter III we present an analytical approach to the solution for minimum time, lateral turns in a horizontal plane of a lifting missile. To simplify the problem, it is assumed that the thrust direction is always aligned with the velocity vector. The lift and the bank controls and the thrusting program for the three types of optimal arcs involved are expressed in terms of the state variables and a set of constants of integration. Hence, the variational problem is resolved ultimately to a multi-point boundary-value problem. For the case of rectilinear flight, and for the case of turning flight with the final position being free, it is shown that the variable thrust magnitude is given by (see Table of Nomenclature for the notation)

$$\frac{T}{\frac{1}{2}\rho S c^2 C_{D_0}} = f\left(\frac{V}{c}, a\right) \tag{I.1}$$

where f is a known function of V/c and a constant of integration "a". Furthermore, the function f is insensitive to "a". Hence this formula displays explicitly the variation of T in terms of the flight velocity, or equivalently, as a function of Mach number. We notice that T is proportional to the zero lift drag coefficient C_{D_0}, and the atmospheric mass density ρ. Hence, the variable thrust magnitude for sustaining flight is an exponential function of the altitude. This formula gives a complete criterion for the programming of the variable thrust profile.

Chapter IV gives the numerical results of the same problem. In this chapter a discussion of the computer program used to compute solutions to fully constrained problems is presented along with simulation results for representative cases. While in Chapter III, to ease the analytical discussion, it is assumed that the aerodynamic characteristics C_{D_0} and k in the parabolic drag polar representation

$$C_D = C_{D_0}(M) + k(M)C_L^2 \tag{I.2}$$

were independent of the Mach number, the numerical program discussed in Chapter IV has provision for inclusion of these variations.

In the concluding chapter, Chapter V, we discuss the five types of rocket motors used in Air Force missions in order of increasing thrust controllability. A qualitative analysis was carried out for some typical missions. The numerical simulations indicate that the percentage performance gain with throttling increases with the difficulty of the mission (especially position target missions).

II. OPTIMAL THRUST MAGNITUDE CONTROL

Optimal thrust magnitude control for flight in a vacuum has been discussed by Leitmann (Ref. 1) and Marec (Ref. 2). For flight inside an atmosphere, partial results mainly concerning a rocket engine with constant exhaust velocity mounted rigidly fixed to the vehicle flying in a flat earth model have been obtained by several authors (e.g., Bryson and Lele (Ref. 3)).

In this chapter we extend the results of Marec concerning a general propulsion system to the case of flight in a general force field and in a resisting medium.

II.1 The Propulsion Systems

Let $\vec{T}$ be the thrust developed by the engine. It is assumed that the direction of the thrust can be taken arbitrarily. It will be shown that in this case the optimal thrust direction is parallel to the vector $\vec{p}_V$, associated to the velocity vector. The two remaining control parameters are the thrust magnitude T and the mass flow $\beta = -\frac{dm}{dt}$.

Let us define a general propulsion system (S) in the (T, β) space (Fig. II.1). Let c be the exhaust velocity of the gas ejected from the engine. The thrust magnitude is then

$$T = \beta c \tag{II.1}$$

The power of the propulsion system is

$$P = \frac{1}{2}(-\dot{m})c^2 = \frac{Tc}{2} = \frac{T^2}{2\beta} \tag{II.2}$$

The power being limited by

$$P \leqq P_{max} \tag{II.3}$$

the control domain in the (T, β) space is bounded by the parabola OABC given by the equation

$$T^2 = 2P_{max}\beta \tag{II.4}$$

In the general system, system (S), we assume that the exhaust velocity is bounded by

$$c_{min} \leqq c \leqq c_{max} \tag{II. 5}$$

This further restricts the control domain in the (T,β) space. In general the mass flow is bounded by

$$\beta \leqq \beta_{max} \tag{II. 6}$$

and the resulting control space is shown in Fig. II. 1.

We notice that the thrust magnitude is bounded (Fig. II. 1)

$$T \leqq T_{max} = \min(T_B, T_c) \tag{II. 7}$$

From the general propulsion system (S) we have the following special cases:

If the exhaust velocity is constant, $c_{min} = c_{max} = c$, we have the subsystem (S_1) (Fig. II. 2). In this system, the only control parameter left besides the thrust direction is either the mass flow β, or the thrust magnitude T. The thrust is bounded by

$$0 \leqq T \leqq T_{max} \tag{II. 8}$$

For an ideal electric propulsion system, the subsystem (S_2) (Fig. II. 3), we have

$$c_{min} = 0, \; c_{max} = \infty, \; \beta_{max} = \infty \tag{II. 9}$$

The control space is bounded by the limitation of the power only.

In the following, we shall examine the thrust magnitude control of these models with a detailed analysis of the system (S_1), namely a rocket engine with constant exhaust velocity.

II. 2 Optimal Thrust Magnitude Control

The motion of a vehicle, considered as a mass point, flying in a general gravitational field and subject to aerodynamic and thrusting forces, is governed by the equations

$$\frac{d\vec{r}}{dt} = \vec{V}$$

$$\frac{d\vec{V}}{dt} = \frac{1}{m}(\vec{T} + \vec{A}) + \vec{g}(\vec{r}, t) \qquad \text{(II.10)}$$

$$\frac{dm}{dt} = -\frac{T}{c}$$

Using the maximum principle, for a minimizing problem, we form the Hamiltonian

$$H = \vec{p}_r \cdot \vec{V} + \vec{p}_v \cdot [\frac{1}{m}(\vec{T} + \vec{A}) + \vec{g}] - p_m \frac{T}{c} \qquad \text{(II.11)}$$

where $\vec{p}_r$, $\vec{p}_v$ and p_m are the adjoints to $\vec{r}$, $\vec{V}$ and m.

For optimal thrust control, we maximize H with respect to $\vec{T}$. Consider the vector $\vec{p}_v$, called the primer vector. For the direction of $\vec{T}$, maximizing H is equivalent to selecting $\vec{T}$ such that the dot product $\vec{p}_v \cdot \vec{T}$, is maximized. Hence we have the generalization of Lawden's law for flight in a vacuum, namely that the direction of $\vec{T}$ should be orientated along the vector $\vec{p}_v$ and

$$\max(\vec{p}_v \cdot \vec{T}) = p_v T \qquad \text{(II.12)}$$

where p_v is the magnitude of T. Using this condition and upon replacing c by T/β we rewrite the Hamiltonian (II.11)

$$H = -p_m \beta + \frac{p_v}{m} T + G \qquad \text{(II.13)}$$

where G is independent of the control elements β and T.

We shall consider successively the maximization of H in the control space (T, β) for the different propulsion systems (S), (S_1) and (S_2).

II.2.1 Optimal Thrust Magnitude Control Using the System (S)

At each instant t, the equation H = constant is the equation of a straight line in the plane (T, β) (Fig. II.4). The slope of this straight line is $\frac{mp_m}{p_v}$. The operating point in the (T, β) space depends on the value of this slope. When $\beta = 0$

$$T_{\beta=0} = \frac{m}{p_v}(H - G) \tag{II.14}$$

Since m/p_v is positive, to maximize H we select the operating point ρ in the bounded domain (T, β) representing the system (S) such that $T_{\beta=0}$ is maximum possible. Hence we have the following cases.

1. If $\frac{mp_m}{p_v} > c_{max}$ we should select the point O. $T = 0$ (coasting flight).
2. If $\frac{mp_m}{p_v} = c_{max}$, the operating point can be anywhere in the segment OA. We have T = variable, with $c = c_{max}$. We have the case of sustaining flight. (Singular case)
3. If $\frac{c_{max}}{2} \leqq \frac{mp_m}{p_v} < c_{max}$, by noticing that $\frac{c_{max}}{2}$ is the slope of the tangent to the parabola AB at the point A, we deduce that the maximum of H is obtained by using the point A. The thrust magnitude is constant (Boosting flight at constant thrust)

$$T = T_A = \frac{2P_{max}}{c_{max}} \tag{II.15}$$

The corresponding mass flow is

$$\beta = \frac{2P_{max}}{c^2_{max}} \tag{II.16}$$

4. If $\max\left(\frac{C_B}{2}, \frac{C_c}{2}\right) \leqq \frac{mp_m}{p_v} \leqq \frac{c_{max}}{2}$, the operating point is on the arc $\widehat{AB}$ (or $\widehat{AC}$) of the parabola (II.4). The thrust magnitude is variable and is given by

$$T = \frac{P_{max}p_v}{mp_m} \tag{II.17}$$

Hence it is determined by the ratio p_v/mp_m. The corresponding mass flow is

$$\beta = \frac{1}{2}\left(\frac{p_v}{mp_m}\right)^2 P_{max} \tag{II.18}$$

5. If $\frac{mp_m}{p_v} < \max(\frac{C_B}{2}, \frac{C_c}{2})$, the operating point is at the point B (or C). The thrust magnitude is at the maximum constant thrust (Boosting flight at maximum thrust).

Hence for a system (S) we have five types of optimal arcs.

1. $T = 0$ (Coasting Arc)
2. T = variable at low level (Sustaining Arc at low thrust)
3. T = constant at low level (Boosting Arc at constant thrust)
4. T = variable at high level (Sustaining Arc at high thrust)
5. $T = T_{max}$ (Boosting Arc at maximum thrust)

Besides the singular case (case 2) which cannot be treated by first order theory, the thrust magnitude control is governed by the quantity mp_m/p_v.

II.2.2 Optimal Thrust Magnitude Control Using the System (S_2)

The optimal thrust magnitude control for the system (S_2) can be easily deduced from the results concerning the more general system (S). The operating point is always on the parabola given by Eq. (II.4). Hence the thrust magnitude can be taken as the control available, the engine being operated at maximum power for any mass flow. The optimal thrust magnitude is given by

$$T = \frac{p_v}{mp_m} P_{max} \tag{II.19}$$

Hence it is a function of the quantity p_v/mp_m.

II.2.3 Optimal Thrust Magnitude Control Using the System (S_1)

Finally, let us consider the propulsion system represented by the model (S_1). The exhaust velocity being constant, the thrust magnitude can be used as the unique control. Using the same geometric approach as for the system (S) we can easily show the three types of optimal arc

1. If $\frac{mp_m}{p_v} > c$, we take $T = 0$ (coasting arc)

2. If $\frac{mp_m}{p_v} = c$, we take T = variable (sustaining arc). This case is the singular case and the thrust magnitude cannot be decided by the first order theory.

3. If $\frac{mp_m}{p_v} < c$, we take $T = T_{max}$ (boosting arc).

Analytically, using the optimal law (II.12) for the thrust direction, we rewrite (II.11) for this case where c = constant.

$$H = \vec{p}_r \cdot \vec{V} + \vec{p}_v \cdot \vec{g} + \frac{1}{m}(\vec{p}_v \cdot \vec{A}) + \frac{T}{m}(p_v - \frac{mp_m}{c}) \qquad \text{(II.20)}$$

We define the switching function

$$K = p_v - \frac{mp_m}{c} \qquad \text{(II.21)}$$

Then, to maximize H with respect to T:

if $K < 0$, we select $T = 0$

if $K = 0$, we select T = variable

if $K > 0$, we select $T = T_{max}$.

We notice that, if the gravitational field is time invariant, the Hamiltonian H is a constant of the motion.

To derive the expression for the variable thrust magnitude control, we notice that in this case $K = 0$, and we have the relation

$$p_v = \frac{mp_m}{c} \qquad \text{(II.22)}$$

At this point, it is necessary to make assumptions concerning the atmospheric force $\vec{A}$ and the acceleration of the gravitational field $\vec{g}$.

We assume that

$$\vec{A} = \frac{1}{2}\rho S V^2 \vec{a} \qquad \text{(II.23)}$$

where ρ is the atmospheric mass density

$$\rho = \rho_0 e^{-\lambda(r-R)} \qquad \text{(II.24)}$$

where R is a reference length, usually taken as the radius of the planet and λ is the height scale constant. We shall consider a Newtonian gravitational field

$$\vec{g} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{II.25}$$

With these assumptions we have the equations for the adjoints

$$\begin{aligned}
\frac{d\vec{p}_r}{dt} &= \frac{\mu}{r^3}[\vec{p}_v - \frac{3(\vec{p}_v \cdot \vec{r})}{r^2}\vec{r}] + \frac{\lambda}{m}(\vec{p}_v \cdot \vec{A})\frac{\vec{r}}{r} \\
\frac{d\vec{p}_v}{dt} &= -\vec{p}_r - \frac{2(\vec{p}_v \cdot \vec{A})}{mV^2}\vec{V} \\
\frac{dp_m}{dt} &= \frac{1}{m^2}(\vec{p}_v \cdot \vec{A}) + \frac{1}{m^2}p_v T
\end{aligned} \tag{II.26}$$

Now, along a sustaining arc, the equation (II.22) is identically satisfied for a finite time interval. Hence we can take its derivative to have

$$\frac{dp_v}{dt} = \frac{m}{c}\frac{dp_m}{dt} + \frac{p_m}{c}\frac{dm}{dt} = \frac{1}{mc}(\vec{p}_v \cdot \vec{A}) + \frac{T}{mc}(p_v - \frac{mp_m}{c})$$

Using (II.22) we have

$$\frac{dp_v}{dt} = \frac{1}{mc}(\vec{p}_v \cdot \vec{A}) \tag{II.27}$$

On the other hand

$$p_v\frac{dp_v}{dt} = \vec{p}_v \cdot \frac{d\vec{p}_v}{dt} = -\vec{p}_r \cdot \vec{p}_v - \frac{2(\vec{p}_r \cdot \vec{A})(\vec{p}_v \cdot \vec{V})}{mV^2} \tag{II.28}$$

By eliminating dp_v/dt between the last two equations, we have another relation for sustaining flight

$$-\vec{p}_r \cdot \vec{p}_v = \frac{(\vec{p}_v \cdot \vec{A})}{mc}[p_v + \frac{2c(\vec{p}_v \cdot \vec{V})}{V^2}] \tag{II.29}$$

Using the Hamiltonian (II.20) along a sustaining arc we can give this relation another form independent of the aerodynamic force

$$-c(\vec{p}_r \cdot \vec{p}_v) = [H - \vec{p}_r \cdot \vec{V} + \frac{\mu}{r^3}(\vec{p}_v \cdot \vec{r})][p_v + \frac{2c(\vec{p}_v \cdot \vec{V})}{V^2}] \tag{II.30}$$

If the adjoints $\vec{p}_r$ and $\vec{p}_v$ can be found, this equation gives the relationship among the state variables $\vec{r}, \vec{v}$ and m along a sustaining arc. Special cases of this relation, mainly for flat earth model and horizontal flight have been obtained by various authors (e.g., Ref. 3).

We also notice that, if the aerodynamic force is vanishingly small, then from (II.29)

$$\vec{p}_r \cdot \vec{p}_v = 0 \tag{II.31}$$

We have the classical result for orbital transfer in a vacuum, namely along a sustaining arc where the thrust magnitude is variable, the two adjoint vectors $\vec{p}_r$ and $\vec{p}_v$ are orthogonal.

The thrust magnitude along a sustaining arc is obtained by taking the derivative of the singular relation (II.30). After some algebraic manipulation we have

$$\begin{aligned} c\left\{p_r^2 - \frac{\mu}{r^3}\left[p_v^2 - \frac{3(\vec{p}_v \cdot \vec{r})^2}{r^2}\right]\right\} = &- \frac{(\vec{p}_r \cdot \vec{A})}{m}\left[p_v + \frac{2(\vec{p}_v \cdot \vec{V})}{V^2} c\right] \\ &+ \frac{(\vec{p}_v \cdot \vec{A})}{m}\left\{- \frac{4Hc}{V^2} + \frac{(\vec{p}_v \cdot \vec{A})}{mc}\left(1 + \frac{2c^2}{V^2}\right) + \frac{c(\vec{p}_v \cdot \vec{r})}{r}\left(\lambda - \frac{6\mu}{r^2 V^2}\right) - \frac{4c(\vec{p}_v \cdot \vec{V})(\vec{A} \cdot \vec{V})}{mV^4}\right. \\ &\left. - \frac{(\vec{r} \cdot \vec{V})}{r}\left[\frac{2\lambda c(\vec{p}_v \cdot \vec{V})}{V^2} + p_v\left(\lambda + \frac{2\mu}{r^2 V^2}\right)\right]\right\} + \frac{(\vec{p}_v \cdot \vec{A})}{m^2}\left[\frac{p_v}{c}\left(1 + \frac{2c^2}{V^2}\right) + \frac{4(\vec{p}_v \cdot \vec{V})}{V^2}\right] T \end{aligned} \tag{II.32}$$

We see that if $(\vec{p}_v \cdot \vec{A}) \neq 0$, the variable thrust magnitude can be obtained from this relation. Some special cases of this equation, giving T explicitly in terms of the dimensionless velocity V/c will be presented in the next chapter.

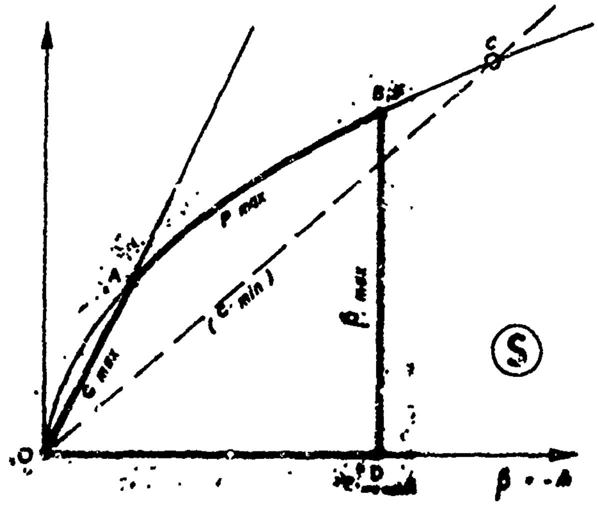


Fig. II. 1 Propulsion System (S)

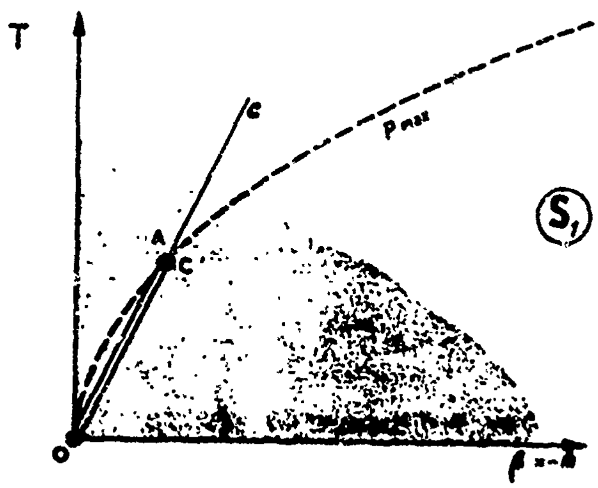


Fig. II. 2 Propulsion System (S_1)
(c = constant)

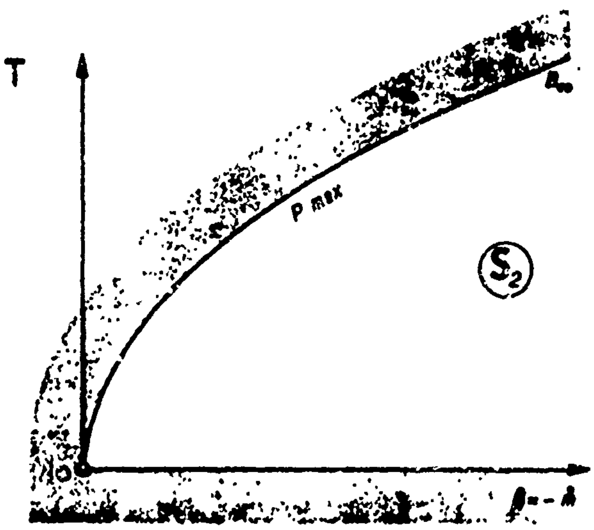


Fig. II. 3 Propulsion System (S_2)

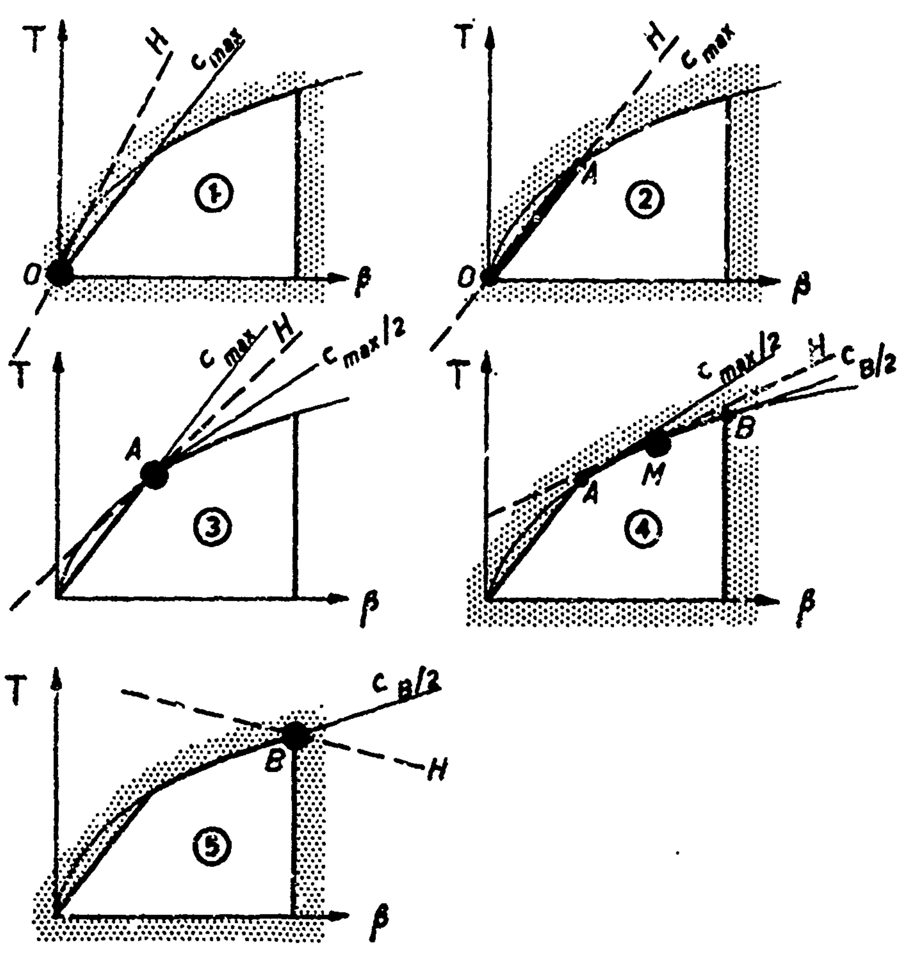


Fig. II. 4 Optimal Thrust Magnitude Control for System (S)

III. ANALYTICAL SOLUTIONS

In this chapter we present an analytical approach to the solution for minimum time, lateral turns in a horizontal plane of a lifting missile. The lift and the bank controls and the thrusting program for the three types of optimal arcs involved will be expressed in terms of the state variables and a set of constants of integration. Hence, the variational problem is resolved ultimately to a multi-point boundary-value problem.

Simple cases where certain end-conditions are relaxed can be solved analytically. Furthermore, in these cases, the variable thrust profiles are obtained explicitly, thus allowing constructive suggestions for the design of TMC rockets.

III. 1. Equations of Flight in a Horizontal Plane

The equations of motion for a coordinated turn in a horizontal plane, with the thrust always aligned with the velocity, are (Fig. III 1)

$$
\begin{aligned}
\frac{dX}{dt'} &= V \cos\beta \\
\frac{dY}{dt'} &= V \sin\beta \\
\frac{dV}{dt'} &= \frac{(T - D)}{m} \\
V\frac{d\beta}{dt'} &= \frac{L \sin\sigma}{m} \\
\frac{dm}{dt'} &= -\frac{T}{C}
\end{aligned}
\qquad \text{(III. 1)}
$$

The notation is given in the nomenclature section. Note that we use t' to designate the real time. For the flight to stay in the horizontal plane, we have the constraining relation

$$L \cos\sigma = mg \qquad \text{(III. 2)}$$

The drag polar used is a parabolic drag polar

$$C_D = C_{D_0} + kC_L^2 \qquad \text{(III. 3)}$$

In order to obtain an analytical solution, we shall assume that the zero-lift drag coefficient C_{D_0} and the induced drag factor k are independent of the Mach number and the Reynolds number. This is especially true for flight in the hypervelocity regime. Aerodynamic configurations and flight regimes where these coefficients depend on the Mach number will be considered in the next chapter where a complete numerical solution for a specified type of missile will be presented. We shall use the usual assumption for the lift and drag forces of the form

$$L = \frac{1}{2}\rho S C_L V^2$$
$$D = \frac{1}{2}\rho S C_D V^2 \qquad \text{(III. 4)}$$

To obtain general optimal laws for a general type of missile we introduce the nondimensional quantities

$$t = \frac{gt'}{c}, \quad x = \frac{gX}{c^2}, \quad y = \frac{gY}{c^2}$$

$$w = \frac{mg}{\frac{1}{2}\rho S c^2 \sqrt{C_{D_0}/k}}, \quad \tau = \frac{T}{\frac{1}{2}\rho S c^2 C_{D_0}} \qquad \text{(III. 5)}$$

$$u = \frac{V}{c}, \quad \eta = \sqrt{kC_{D_0}} = \frac{1}{2E^*}$$

If the angle-of-attack, or equivalently the lift coefficient C_L, is not constrained, a natural choice for the aerodynamic control would be the bank angle σ. Then, the lift and the drag coefficients will be obtained from (III. 2) and (III. 3). In the practical case where the lift coefficient is bounded, the corresponding bound on the bank angle is a function of the state variables m and V through the relation (III. 2). Hence, the lift coefficient will be a better choice as control parameter in this case.

We define a lift control parameter λ such that λ^2 is the ratio of the induced drag to the zero-lift drag.

$$\lambda^2 = \frac{kC_L^2}{C_{D_0}} \qquad \text{(III. 6)}$$

Then we have

$$C_L = \sqrt{C_{D_0}/k}\ \lambda, \quad C_D = C_{D_0}(1+\lambda^2) \tag{III.7}$$

$$\cos\sigma = \frac{w}{\lambda u^2}, \quad \sin\sigma = \frac{\sqrt{\lambda^2 u^4 - w^2}}{\lambda u^2}, \quad \tan\sigma = \frac{\sqrt{\lambda^2 u^4 - w^2}}{w}$$

We notice that when $\lambda = 1$, the flight is effectuated at maximum lift-to-drag ratio E*.

The independent controls are τ and λ. They are bounded in the space

$$\begin{aligned} 0 &\leqq \tau \leqq \tau_M \\ 0 &\leqq \lambda \leqq \lambda_M \end{aligned} \tag{III.8}$$

The state equations become, with the nondimensional variables,

$$\begin{aligned}
\frac{dx}{dt} &= u\cos\beta \\
\frac{dy}{dt} &= u\sin\beta \\
\frac{du}{dt} &= \frac{\eta}{w}[\tau - (1+\lambda^2)u^2] \\
\frac{d\beta}{dt} &= \frac{\sqrt{\lambda^2 u^4 - w^2}}{uw} \\
\frac{dw}{dt} &= -\eta\tau \\
\frac{dt}{dt} &= 1
\end{aligned} \tag{III.9}$$

To write the variational equations for optimal trajectories, we introduce the adjoint components p_i, $(i = 0,\dots,5)$, to form the Hamiltonian

$$H = p_0 + u(p_1\cos\beta + p_2\sin\beta) - \frac{\eta p_3 (1+\lambda^2)u^2}{w} + \frac{p_4\sqrt{\lambda^2 u^4 - w^2}}{uw} + \frac{\eta}{w}\tau(p_3 - wp_5) \tag{III.10}$$

The p_i, $(i = 1,\dots,5)$, are respectively paired with the first five equations (III.9) while p_0 is associated to the last equation. These adjoint variables satisfy the equations

$$\frac{dp_0}{dt} = 0$$

$$\frac{dp_1}{dt} = 0$$

$$\frac{dp_2}{dt} = 0$$

$$\frac{dp_3}{dt} = -(p_1 \cos\beta + p_2 \sin\beta) + \frac{2\eta(1+\lambda^2)u}{w} p_3 - \frac{(\lambda^2 u^4 + w^2)}{wu^2\sqrt{\lambda^2 u^4 - w^2}} p_4$$

$$\frac{dp_4}{dt} = u(p_1 \sin\beta - p_2 \cos\beta) \qquad \text{(III.11)}$$

$$\frac{dp_5}{dt} = \frac{\eta\tau p_3}{w^2} - \frac{\eta(1+\lambda^2)u^2}{w^2} p_3 + \frac{\lambda^2 u^3}{w^2\sqrt{\lambda^2 u^4 - w^2}} p_4$$

The solution to the problem is obtained by integrating the system of state equations (III.9), and adjoint equations (III.11) with the appropriate end-conditions, while selecting τ and λ, subject to the constraints (III.8), in such a way that, at each instant, the Hamiltonian H given by (III.10) is an absolute maximum.

III.2. Optimal Controls

The problem, as formulated, has a number of integrals of the motion. First, H does not contain t explicitly. Since the final time is not specified

$$H \equiv \text{constant} = 0 \qquad \text{(III.12)}$$

we see that p_0 = constant, and for minimum time problem

$$p_0 = -1 \qquad \text{(III.13)}$$

Also, from the adjoint equations

$$p_1 = a_1 = \text{constant}$$
$$p_2 = a_2 = \text{constant} \qquad \text{(III.14)}$$

Using these relations we write the equation for p_4

$$\frac{dp_4}{dt} = a_1\frac{dy}{dt} - a_2\frac{dx}{dt} = \frac{d}{dt}(a_1 y - a_2 x)$$

Upon integrating

$$p_4 = a_1 y - a_2 x + a_4 \qquad \text{(III.15)}$$

where a_4 is a constant. We notice that the constants obtained thus far are valid over the entire optimal trajectory

By inspection of the Hamiltonian, we define the switching function

$$K = p_3 - wp_5 \tag{III.16}$$

Then, for the thrust magnitude control, to maximize H with respect to τ:

if

$K > 0$, we select $\tau = \tau_M$ (boosting phase)

$K < 0$, we select $\tau = 0$ (coasting phase) (III.17)

$K = 0$, for a finite time interval,

we select τ = variable (sustaining phase)

For the lift control, considered as function of λ, H reaches a maximum either at

$$\lambda = \lambda_M \tag{III.18}$$

or at an interior point given by $\frac{\partial H}{\partial \lambda^2} = 0$. Explicitly, we have

$$\sqrt{\lambda^2 u^4 - w^2} = \frac{p_4 u}{2\eta p_3} \tag{III.19}$$

Hence, if the maximum lift coefficient is used, we have for the bank, the maximum possible bank angle for horizontal flight

$$\cos\sigma = \frac{w}{\lambda_M u^2} \tag{III.20}$$

In the case where variable lift coefficient is used, the bank angle is given by

$$\tan\sigma = \frac{p_4 u}{2\eta p_3 w} \tag{III.21}$$

We shall refer to the interior lift and bank as the normal lift and normal bank controls. The other optimal possibilities will be referred to as the maximum lift and the maximum bank controls. We shall assume that there are no wavering maneuvers, the heading being continuously increasing along a turning flight, and hence $\sigma > 0$. This in turn implies that p_3 and p_4 are of the same sign.

We shall analyze separately each of the three types of optim . arcs

III-2-1 Coasting Arc

Let

$$\Lambda = \sqrt{\lambda^2 u^4 - w^2} \qquad \text{(III.22)}$$

$$A = a_1 \cos\beta + a_2 \sin\beta$$

Along a coasting arc, $\tau = 0$, and we write the Hamiltonian

$$uA = 1 + \frac{\eta(1+\lambda^2)u^2 p_3}{w} - \frac{\Lambda p_4}{uw} \qquad \text{(III.23)}$$

The lift control is either

$$\lambda = \lambda_M \quad \text{or} \quad \Lambda = \frac{p_4 u}{2\eta p_3} \qquad \text{(III.24)}$$

We first consider the case where $\lambda = \lambda_M$. The equations of motion, with u as the new independent variable are

$$w = \text{constant}$$

$$\frac{dx}{du} = -\frac{w}{\eta(1+\lambda_M^2)} \frac{\cos\beta}{u}$$

$$\frac{dy}{du} = -\frac{w}{\eta(1+\lambda_M^2)} \frac{\sin\beta}{u} \qquad \text{(III.25)}$$

$$\frac{d\beta}{du} = -\frac{\sqrt{\lambda_M^2 u^4 - w^2}}{\eta(1+\lambda_M^2)u^3}$$

$$\frac{dt}{du} = -\frac{w}{\eta(1+\lambda_M^2)u^2}$$

The boundary conditions at the ends of the coasting arc are

$$t = t_0,\ u = u_0,\ x = x_0,\ y = y_0,\ \beta = \beta_0,\ w = w_0$$

$$t = t_1,\ u = u_1,\ x = x_1,\ y = y_1,\ \beta = \beta_1,\ w = w_0 \qquad \text{(III.26)}$$

By integrating the last equation (III.25) we have for the velocity

$$\frac{w_0}{\eta(1+\lambda_M^2)}\left(\frac{1}{u} - \frac{1}{u_0}\right) = t - t_0 \qquad \text{(III.27)}$$

The change in the velocity from the initial point is obtained by replacing u and t by u_1 and t_1 in this equation.

The variation in the heading is given by

$$2\eta(1+\lambda_M^2)(\beta-\beta_0) = \lambda_M \log \frac{\lambda_M u_0^2 + \sqrt{\lambda_M^2 u_0^4 - w_0^2}}{\lambda_M u^2 + \sqrt{\lambda_M^2 u^4 - w_0^2}} + \frac{\sqrt{\lambda_M^2 u^4 - w_0^2}}{u^2}$$

$$- \frac{\sqrt{\lambda_M^2 u_0^4 - w_0^2}}{u_0^2} \qquad \text{(III.28)}$$

The change in heading is obtained by replacing β and u by β_1 and u_1 in this equation.

For the trajectory, it is simpler to use the arc length. By the transformation

$$dx = \cos\beta\, ds$$
$$dy = \sin\beta\, ds \qquad \text{(III.29)}$$

we have

$$ds = -\frac{w_0}{\eta(1+\lambda_M^2)} \frac{du}{u} \qquad \text{(III.30)}$$

Upon integrating

$$s - s_0 = \frac{w_0}{\eta(1+\lambda_M^2)} \log \frac{u_0}{u} \qquad \text{(III.31)}$$

where $s - s_0$ is the arc length travelled since the initial time.

Next we consider the case of normal lift control. By eliminating p_3 between (III.23) and the second equation (III.24) we have

$$uA = 1 - \frac{p_4}{2uw\Lambda}[\Lambda^2 - (u^4 + w^2)] \qquad \text{(III.32)}$$

The state equations along a coasting arc become

$$w = \text{constant}$$

$$\frac{dx}{du} = -\frac{wu^3 \cos\beta}{\eta[\Lambda^2 + (u^4 + w^2)]}$$

$$\frac{dy}{du} = - \frac{wu^3 \sin\beta}{\eta[\Lambda^2 + (u^4 + w^2)]} \qquad \text{(III.33)}$$

$$\frac{d\beta}{du} = - \frac{u\Lambda}{\eta[\Lambda^2 + (u^4 + w^2)]}$$

$$\frac{dt}{du} = - \frac{wu^2}{\eta[\Lambda^2 + (u^4 + w^2)]}$$

Since p_4 is given in terms of x and y by (III.15), Eq. (III.32) can be solved for Λ in terms of the state variables x, y, u and β. Subsequently the first three equations (III.33) can be integrated numerically. By satisfying the terminal conditions at the ends of the coasting arc one can deduce the values of the constants a_1, a_2 and a_4 and the three new constants of integrations b_1, b_2 and b_4 arising from the integration. The minimum time of flight is obtained by integrating the last equation (III.33).

Hence, for a pure coasting flight, when the final states x, y, u and β are all constrained, the optimal flight is at variable angle of attack. On the other hand, when these values are free, the flight is effected at maximum angle of attack.

We notice that if the optimal trajectory also includes other types of arcs, then the constants of integration a_1, a_2 and a_4 remain the same throughout the entire trajectory, while the constants of integration b_1, b_2 and b_4 are functions of the values of the state variables at the ends of the coasting arc. Their determination will specify the switching point joining a coasting arc and a thrusting arc.

III-2-2 <u>Sustaining Arc</u>

Along a sustaining arc, K = 0, and we have

$$p_3 - wp_5 = 0 \qquad \text{(III.34)}$$

Consequently, relation (III.23) is also valid along this type of arc.

We first consider the case where this arc is flown with maximum lift. Consider, in general

$$\frac{d(wp_5)}{dt} = \frac{\eta\tau}{w}(p_3 - wp_5) - \frac{\eta(1+\lambda^2)u^2}{w} p_3 + \frac{\lambda^2 u^3}{w\Lambda} p_4$$

Along a coasting arc, $\tau = 0$, and a sustaining arc, $K = 0$, we have

$$\frac{d(wp_5)}{dt} = -\frac{\eta(1+\lambda^2)u^2}{w} p_3 + \frac{\lambda^2 u^3}{w\Lambda} p_4 \tag{III.35}$$

On the other hand, from the adjoint equations (III.11)

$$\frac{dp_3}{dt} = -A + \frac{2\eta(1+\lambda^2)}{w} up_3 - \frac{(\lambda^2 u^4 + w^2)}{wu^2\Lambda} p_4 \tag{III.36}$$

By taking the derivative of (III.34) and using (III.35) and (III.36) we have

$$A = \frac{\eta(1+\lambda^2)u(u+2)}{w} p_3 - \frac{\lambda^2 u^4 (u+1) + w^2}{wu^2\Lambda} p_4 \tag{III.37}$$

By eliminating p_3 between (III.23) and (III.37), and putting $\lambda = \lambda_M$, we have a relation among the state variables along a sustaining arc flown at maximum lift

$$uw\Lambda[(u+2) - u(u+1)A] = (a_1 y - a_2 x + a_4)[\Lambda^2 - w^2(u+2)] \tag{III.38}$$

Upon taking the derivative of this equation, using the state equations for simplification we can express the variable thrust magnitude control in terms of the state variables. Explicit expressions for the thrust profile in some special cases of interest will be presented in the next section.

For the case where the sustaining arc is flown with variable lift coefficient, we use the second relation (III.24) to rewrite (III.23) and (III.37) we have

$$uA = 1 - \frac{[\Lambda^2 - (u^4 + w^2)]}{2uw\Lambda} p_4 \tag{III.39}$$

and

$$A = \frac{p_4}{2uw\Lambda}\left[\frac{(u+2)(u^4 - w^2)}{u} - \Lambda^2\right] \tag{III.40}$$

The relation among the state variables along a sustaining arc using variable lift is obtained by eliminating the lift control Λ between these two equations. We have for the variable lift control

$$\Lambda^2 = \frac{(u+2)(u^4 - w^2) - u[u^4(u+1) - (u+3)w^2]A}{u[1+(1-u)A]} \tag{III.41}$$

or explicitly

$$\lambda^2 = 1 + \frac{2[(u^4 - w^2) - u(u^4 - 2w^2)A]}{u^5[1+(1-u)A]} \tag{III.42}$$

On the other hand we have, by eliminating Λ^2 between (III.39) and (III.40)

$$\Lambda = \frac{p_4[u^4 - (u+1)w^2]}{u^2 w[1+(1-u)A]} \tag{III.43}$$

From the equations (III.41) and (III.43) we have the relation among the state variables along a sustaining arc, using variable lift control.

$$p_4^2[u^4 - (u+1)w^2]^2 = u^3w^2[1+(1-u)A]\{(u+2)(u^4 - w^2) - u[u^4(u+1) - (u+3)w^2]A\} \tag{III.44}$$

The variable thrust magnitude control will appear upon taking the derivative of this equation.

III-2-3 Boosting Arc

An explicit formula for the normal lift along a boosting arc in terms of the state variables only is not known analytically. For the case where the boosting arc is flown with maximum lift coefficient, we have the equation

$$\frac{du}{dw} = -\frac{1}{w}(1 - au^2) \tag{III.45}$$

where a is a constant

$$a = \frac{1+\lambda_M^2}{\tau_M} \tag{III.46}$$

By integrating we have the relation between the mass and the velocity along a boosting arc flown at maximum lift

$$w\left(\frac{1+\sqrt{a}\,u}{1-\sqrt{a}\,u}\right)^{\frac{1}{2\sqrt{a}}} = C \tag{III.47}$$

where C is a constant. When $\tau_M \to \infty$, $a \to 0$, we write $b = 1/2\sqrt{a}$ and consider

$$\left(\frac{1+\sqrt{a}\;u}{1-\sqrt{a}\;u}\right)^{\frac{1}{2\sqrt{a}}} = \left[\left(1+\frac{u}{b}\right)+O\left(\frac{1}{b^2}\right)\right]^{b} \rightarrow e^{u} \text{ as } b \rightarrow \infty$$

Hence, if impulsive thrust is permitted, we have, along an impulsive-thrust arc

$$we^{u} = C \tag{III.48}$$

III.3. Solutions for some Special Cases

We have shown in the preceding section that for the general case, the optimal controls can be expressed explicitly in terms of the state variables and a certain number of constants of integration. In other words, the set of adjoint equations, except for the case of maximum thrust with variable lift coefficient, can be integrated completely. Thus the variational problem is reduced to a multi-point boundary-value problem. Because of the non-linearity of the state equations, the actual optimal trajectory can be obtained only by using numerical integration.

In this section we shall relax some final conditions on the state variables in order to get deeper into the problem analytically. In particular we are able to obtain the thrust profile along a sustaining arc in these cases. This result, coupled with the numerical investigation in the next chapter will allow us to formulate constructive suggestions for the design of TMC rockets.

III-3-1 Rectilinear Flight

This problem has been analyzed numerically in Ref.4, for a very special type of rocket. In this section we shall give the general analytical solution to the problem for a general type of lifting rocket.

For this case $y = 0$, $\beta = 0$, $\sigma = 0$, and the lift coefficient is always

$$\lambda = \frac{w}{u^2} \leq \lambda_M \tag{III.49}$$

The lift coefficient is no longer an independent control, and we should rewrite the two sets of state, and adjoint equations using the thrust magnitude as the unique control.

For the state equations we have

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = \frac{\eta}{w}\left[\tau - \frac{(w^2 + u^4)}{u^2}\right] \tag{III.50}$$

$$\frac{dw}{dt} = -\eta\tau$$

$$\frac{dt}{dt} = 1$$

Using the same subscript order for the adjoint variables as associated with the full set of state variables, we have for the Hamiltonian

$$H = -1 + p_1 u - \frac{\eta(w^2 + u^4)}{wu^2} p_3 + \frac{\eta\tau}{w}(p_3 - wp_5) \tag{III.51}$$

The adjoint equations, in this case, are

$$\frac{dp_1}{dt} = 0$$

$$\frac{dp_3}{dt} = -p_1 + \frac{2\eta(u^4 - w^2)}{wu^3} p_3 \tag{III.52}$$

$$\frac{dp_5}{dt} = \frac{\eta\tau p_3}{w^2} - \frac{\eta(u^4 - w^2)}{w^2 u^2} p_3$$

The switching function is always

$$K = p_3 - wp_5 \tag{III.53}$$

As before, we have the integrals of motion

$$p_1 = a_1 \tag{III.54}$$

and

$$H = 0 \tag{III.55}$$

This gives the relation

$$\frac{\eta(w^2 + u^4)}{wu^2} p_3 = -1 + a_1 u + \frac{\eta\tau}{w}(p_3 - wp_5) \tag{III.56}$$

which, for the cases of coasting flight, and sustaining flight, is reduced to

$$\frac{\eta(w^2 + u^4)}{wu^2} p_3 = -1 + a_1 u \quad \text{(III. 57)}$$

Next, we consider

$$\frac{d(wp_5)}{dt} = \frac{\eta\tau}{w}(p_3 - wp_5) - \frac{\eta(u^4 - w^2)}{wu^2} p_3 \quad \text{(III. 58)}$$

For coasting and for sustaining flight, we have

$$\frac{d(wp_5)}{dt} = -\frac{\eta(u^4 - w^2)}{wu^2} p_3 \quad \text{(III. 59)}$$

We consider succesively the following three types of optimal arc.

<u>Coasting Arc</u>

We have, $\tau = 0$, and hence

$$w = \text{constant} \quad \text{(III. 60)}$$

The adjoint variable p_3 is obtained in terms of u from (III. 57). For p_5, we rewrite (III. 58)

$$\frac{dp_5}{dt} = -\frac{\eta(u^4 - w^2)}{w^2 u^2} p_3 = -\frac{(u^4 - w^2)(a_1 u - 1)}{w(u^4 + w^2)} \quad \text{(III. 61)}$$

The equation for u, with $\tau = 0$ is

$$\frac{du}{dt} = -\frac{\eta(u^4 + w^2)}{wu^2} \quad \text{(III. 62)}$$

Hence

$$\frac{dp_5}{du} = \frac{u^2(a_1 u - 1)(u^4 - w^2)}{\eta(u^4 + w^2)^2} \quad \text{(III. 63)}$$

Since w is constant, p_5 is obtained by quadrature and the system of adjoint equations is completely integrable.

The velocity is obtained by integrating (III. 62) from the initial point of the coasting arc. We have

$$\frac{\eta}{\sqrt{w}}(t - t_0) = \frac{1}{4\sqrt{2}} \log \frac{u^2 - \sqrt{2w}\, u + w}{u^2 + \sqrt{2w}\, u + w} + \frac{1}{2\sqrt{2}} \text{Arc tan} \frac{\sqrt{2w}\, u}{w - u^2} \Bigg|_{u}^{u_0} \quad \text{(III. 64)}$$

For the range, using u as variable, we have

$$\frac{dx}{du} = - \frac{wu^3}{\eta(u^4 + w^2)}$$

Upon integrating

$$\frac{4\eta}{w}(x - x_0) = \log \frac{u_0^4 + w^2}{u^4 + w^2} \qquad \text{(III.65)}$$

<u>Sustaining Arc</u>

Along a sustaining arc, K = 0 and we have

$$p_3 - wp_5 = 0 \qquad \text{(III.66)}$$

By taking the derivative of this equation, and using (III.59) and the second of the Eqs (III.52) we have

$$a_1 = \frac{\eta(u + 2)(u^4 - w^2)}{wu^3} p_3 \qquad \text{(III.67)}$$

By eliminating p_3 between (III.57) and (III.67) we have the relation between the nondimensional weight and the nondimensional velocity along a sustaining arc.

$$\frac{u[u^4(u + 1) - w^2(u + 3)]}{(u + 2)(u^4 - w^2)} = \frac{1}{a_1} \qquad \text{(III.68)}$$

In Fig. III.2, we have plotted the trajectories for sustaining flight, in the (w, u) space, using $a = 1/a_1$ as parameter. Explicitly, we have the equation of the curves with $a < 0$

$$w = u^2 \sqrt{\frac{u^2 + (1 - a)u - 2a}{u^2 + (3 - a)u - 2a}} \qquad \text{(III.69)}$$

The family of curves is bounded by two limiting curves. When $-a \rightarrow \infty$ we have the parabola

$$w = u^2 \qquad \text{(III.69a)}$$

When $a \rightarrow 0$, we have

$$w = u^2 \sqrt{\frac{u + 1}{u + 3}} \qquad \text{(III.69b)}$$

This limiting curve corresponds to the singular curve obtained in Ref. 5, for the problem of maximizing the range.

To have the thrust profile along a sustaining arc, we take the time derivative of (III.69) using the appropriate relations for du/dt and dw/dt. Upon simplification we have

$$\tau = \frac{2u^2[u^2+(2-a)u-2a]}{[u^2+(3-a)u-2a]} \times \qquad \text{(III.70)}$$

$$\frac{[2u^4+(9-4a)u^3+2(a^2-8a+3)u^2-2a(7-4a)u+8a^2]}{[u^5+2(3-a)u^4+(a^2-12a+12)u^3+6(a^2-4a+1)u^2+2a(6a-7)u+8a^2]}$$

In Fig.III.3, we have plotted the thrust magnitude of sustaining flight as a function of the velocity, using "a" as the parameter. The family of curves is bounded by two limiting curves. When $a \to \infty$ we have

$$\tau = \frac{4u^2}{u+2} \qquad \text{(III.70a)}$$

When $a \to 0$, we have

$$\tau = \frac{2u^2(u+2)(2u^2+9u+6)}{(u+3)(u^3+6u^2+12u+6)} \qquad \text{(III.70b)}$$

From Fig.III.2, it is seen that, along a sustaining arc, as w decreases, the velocity decreases. From Fig.III.3, we see that the variable thrust is also decreasing along a sustaining arc. The level of the thrust is high for large velocity, and remains at lower level for small velocity. In particular, for the range of velocity studied in Ref.4, between M = 0.3 and M = 2.5, one easily verifies in our plot for the thrust profile that τ remains small and is slowly decreasing. It has been found numerically in Ref.4 that the thrust magnitude along a sustaining arc is nearly constant and remains small.

We notice that the exact solution for our thrust profile applies to a general type of missile. By the definition (III.5) of τ, we see that the real thrust is proportional to $(\rho S c^2 C_{D_0})$. Hence our formula (III.70) gives explicitly the variable thrust profile in terms of the altitude (through ρ), the flight velocity V, and the missile characteristics S and C_{D_0}.

Using the expression (III.70) for the variable thrust, we have the equation for the distance travelled along a sustaining arc

$$-4\eta(x - x_0) = \int_{u_0}^{u} \frac{2(u+2)ND + 2u(u^2+2a)}{\sqrt{ND}\,[u^2 + (2-a)u - 2a]}\,du \tag{III.71}$$

where

$$\begin{aligned} N &= u^2 + (1-a)u - 2a \\ D &= u^2 + (3-a)u - 2a \end{aligned} \tag{III.72}$$

It is known that the integral in Eq. (III.71) can be expressed in terms of elliptic functions. The time is finally obtained from

$$-4\eta(t - t_0) = \int_{u_0}^{u} \frac{2(u+2)ND + 2u(u^2+2a)}{u\sqrt{ND}\,[u^2 + (2-a)u - 2a]}\,du \tag{III.73}$$

Hence, the equations along a sustaining arc are completely integrable.

Boosting Arc

Along a boosting arc, where $\tau = \tau_M$, the variation in the nondimensional weight is simply

$$w - w_0 = -\eta\tau_M(t - t_0) \tag{III.74}$$

In the (w, u) space, the trajectory for boosting arc is obtained by integrating the equation

$$\frac{du}{dw} = -\frac{1}{w}\left[1 - \frac{(w^2 + u^4)}{\tau_M u^2}\right] \tag{III.75}$$

Although an analytical solution to this equation is not known, it is seen that for $\tau_M \to \infty$, the limiting curve is

$$we^u = C \qquad \text{(I!.}$$

where C is a constant of integration.

Optimal Trajectory

The problem of minimum time between two given end points in the (w,u,x) space is hence obtained by joining the different types of optimal arcs.

Let B = Boosting arc, C = Coasting arc and S = Sustaining arc. Following the same type of discussion as in Ref. 6, we see that there are six possible types of optimal trajectory, namely (Fig. III. 4)

BC		CB
BSC	and	CSB
BSB		CSC

Although the actual computation involves numerical solution, it is possible to have the following general properties which facilitate the computation (Fig. III. 5).

Proposition 1

Let $\vec{X} = (w, u, x)$.

If $\vec{X}_0$ and $\vec{X}_f$ are both specified, then in general the optimal trajectory involves all three types of arc.

Proposition 2

If $w_0 > u_0^2$

the initial arc is a boosting arc

If $w_0 < u_0^2 \sqrt{\dfrac{u_0 + 1}{u_0 + 3}}$

the initial arc is a coasting arc.

Proposition 3

If $w_f > u_f^2$

the final arc is a coasting arc.

If $w_f < u_f^2 \sqrt{\dfrac{u_f + 1}{u_f + 3}}$

the final arc is a boosting arc.

III-3-2 Turning Flight with Free End-point

As in the case for rectilinear flight, the turning flight problem reduces to a problem containing one single parameter if we do not specify the end position x_f and y_f.

In this case, from the general results obtained in Section III-2 we have

$$a_1 = 0, \quad a_2 = 0 \tag{III.77}$$

and hence

$$p_4 = a_4 = \text{constant} \tag{III.78}$$

The controls involved are the lift (and hence the bank angle) and the thrust magnitude. For the optimal lift, we have either

$$\lambda = \lambda_M \tag{III.79}$$

or

$$\sqrt{\lambda^2 u^4 - w^2} = \frac{a_4 u}{2\eta p_3} \tag{III.80}$$

We shall consider the case of maximum lift and next, the case of variable lift.

Maximum Lift Program

The lift is given by (III.79), and hence, we have for the bank angle

$$\cos\sigma = \frac{w}{\lambda_M u^2} \tag{III.81}$$

The optimal arc is of three types.

Coasting Arc

The general solution in section III-2 is valid and we have the following results starting from the initial point w_0, u_0, β_0, s_0.

$$w = w_0$$

$$\frac{\eta(1+\lambda_M^2)}{w_0}(t - t_0) = \frac{1}{u} - \frac{1}{u_0}$$

$$2\eta(1+\lambda_M^2)(\beta - \beta_0) = \lambda_M \log \frac{\lambda_M u_0^2 + \sqrt{\lambda_M^2 u_0^4 - w_0^2}}{\lambda_M u^2 + \sqrt{\lambda_M^2 u^4 - w_0^2}} + \frac{\sqrt{\lambda_M^2 u^4 - w_0^2}}{u^2} - \frac{\sqrt{\lambda_M^2 u_0^4 - w_0^2}}{u_0^2}$$

(III.82)

$$\frac{\eta(1+\lambda_M^2)}{w_0}(s - s_0) = \log\frac{u_0}{u}$$

Sustaining Arc

With $A = 0$, and $\lambda = \lambda_M$, equation (III.38) giving the relation between w and u along a sustaining arc becomes ($a_4 < 0$)

$$\frac{\lambda_M^2 u^4 - w^2(u+3)}{u(u+2)w\sqrt{\lambda_M^2 u^4 - w^2}} = \frac{1}{a_4} \tag{III.83}$$

Explicitly, in terms of u and of the parameter $b = -1/a_4$, we have

$$w = \lambda_M u^2 \sqrt{\frac{[2(u+3)+b^2u^2(u+2)^2]+bu(u+2)\sqrt{(u+2)[4+b^2u^2(u+2)]}}{2[(u+3)^2+b^2u^2(u+2)^2]}} \tag{III.84}$$

Fig. III.6 plots the family of sustaining trajectories in the $((w/\lambda_M), u)$ space, using b as parameter. The curves are bounded by the limiting curves.

For $b \to \infty$, we have

$$w = \lambda_M u^2 \tag{III.84a}$$

For $b \to 0$, we have

$$w = \frac{\lambda_M u^2}{\sqrt{u+3}} \tag{III.84b}$$

The variable thrust along a sustaining arc is obtained by taking the derivative of the singular equation (III.83). We have, after simplification,

$$\begin{aligned}\tau = {} & (1+\lambda_M^2)u^2[3(u+4)+2b^2u^2(u+2)][2(u+3)+b^2u^2(u+2)-bu\sqrt{(u+2)[4+b^2u^2(u+2)]}]/ \\ & [2(u+3)(2u^2+9u+12)+b^2u^2(u+2)(u^3+10u^2+24u+24)+b^4u^4(u+2)^2(u^2+2u+2) \\ & -bu[(u^3+6u^2+12u+12)+b^2u^2(u+2)(u^2+2u+2)]\sqrt{(u+2)[4+b^2u^2(u+2)]}]\end{aligned} \tag{III.85}$$

Fig. III.7 plots the thrust profile $\tau/(1+\lambda_M^2)$ as function of the nondimensional velocity u, using b as parameter. The family of curves is bounded by two limiting curves.

For $b \to \infty$, we have

$$\frac{\tau}{(1+\lambda_M^2)} = \frac{2u^2}{(u+2)} \tag{III.85a}$$

For $b \to 0$, we have

$$\frac{\tau}{(1+\lambda_M^2)} = \frac{3u^2(u+4)}{(2u^2+9u+12)} \tag{III.85b}$$

Once the thrust is known, the variation in the heading, and the time of flight along a sustaining arc are obtained by quadratures. We observe that, along a sustaining arc, both the velocity and the variable thrust level are decreasing.

Boosting Arc

When $\lambda = \lambda_M$, the variation of the mass in terms of the velocity is given by Eq. (III.47) written in terms of the initial conditions at the start of the boosting arc as

$$w\left(\frac{1+\sqrt{a}\,u}{1-\sqrt{a}\,u}\right)^{\frac{1}{2\sqrt{a}}} = w_0\left(\frac{1+\sqrt{a}\,u_0}{1-\sqrt{a}\,u_0}\right)^{\frac{1}{2\sqrt{a}}}, \quad a = \frac{1+\lambda_M^2}{\tau_M} \tag{III.86}$$

The variation of w as function of time is simply

$$w - w_0 = -\eta\tau_M(t - t_0) \tag{III.87}$$

From these we have u as function of t. For the change in the heading, we write

$$\eta\frac{d\beta}{du} = \frac{\sqrt{\lambda_M^2 u^4 - w^2}}{u[\tau_M - (1+\lambda_M^2)u^2]} \tag{III.88}$$

w^2 is obtained from (III.86) and subsequently we have β in terms of u by quadrature.

Normal Lift Program

The variable lift is given by (III.80). The optimal arc using a variable lift coefficient is of three types.

Coasting Arc

Since $A = 0$, $p_4 = a_4$ we rewrite the Eqs. (III.22) - (III.24)

$$\Lambda = \sqrt{\lambda^2 u^4 - w^2}$$

$$1 + \frac{\eta(1+\lambda^2)u^2 p_3}{w} - \frac{a_4\Lambda}{uw} = 0 \tag{III.89}$$

$$\Lambda = \frac{a_4 u}{2\eta p_3}$$

By eliminating p_3 among the equations, we have the equation for Λ.

$$a_4\Lambda^2 - 2uw\Lambda - a_4(u^4 + w^2) = 0 \qquad \text{(III. 90)}$$

Solving for this equation, we have explicitly for the lift coefficient

$$\lambda^2 = 1 + \frac{2w^2}{u^4}\left(1 + \frac{u^2}{a_4^2}\right) \pm \frac{2w}{a_4^2 u^3}\sqrt{u^2 w^2 + a_4^2 (u^4 + w^2)} \qquad \text{(III. 91)}$$

For the bank angle we have ($a_4 \neq 0$)

$$\tan\sigma = \frac{1}{a_4}\left[u \pm \sqrt{u^2 + a_4^2 (1 + \frac{u^4}{w^2})}\right] \qquad \text{(III. 92)}$$

Since w is a constant along a coasting arc, we can plot, for different values of w, the lift coefficient λ, and the bank angle σ, in terms of the flight velocity u, using a_4 as parameter. For a given missile, w can be taken as the initial weight or the final weight. To have the plots valid for a general type of missile, it is convenient to include the constant a_4 in the variables. Let

$$\bar{u} = \frac{u}{a_4}, \quad \bar{w} = \frac{w}{a_4^2} \qquad \text{(III. 93)}$$

Then we have the equation for λ.

$$\lambda^2 = 1 + \frac{2\bar{w}^2}{\bar{u}^4}(1 + \bar{u}^2) \pm \frac{2\bar{w}}{\bar{u}^3}\sqrt{\bar{u}^4 + \bar{w}^2 + \bar{u}^2\bar{w}^2} \qquad \text{(III. 94)}$$

with the condition

$$\frac{\bar{w}^2}{\bar{u}^4} \leq \lambda^2 \leq \lambda_M^2 \qquad \text{(III. 95)}$$

We can have a general plot for the lift coefficient, in terms of the velocity using $\bar{w}$ as parameter.

Similarly, using $\bar{u}$ and $\bar{w}$ we have for the bank angle along a coasting arc using variable lift coefficient

$$\tan\sigma = \bar{u} \pm \sqrt{1 + \bar{u}^2 + \frac{\bar{u}^4}{\bar{w}^2}} \qquad \text{(III. 96)}$$

Since $\bar{w}$ is constant, we see from this formula that when $\bar{u} \to 0$ $\sigma \to 45°$ and when $\bar{u} \to \infty$, $\sigma \to 90°$. Since $\bar{u}$ is decreasing along a coasting arc, the bank angle for normal lift is also decreasing. By Eq. (III. 81) we see that the bank angle is also decreasing along a coasting arc when maximum lift coefficient is used. The analytical results obtained in this chapter are in agreement with the results from an independent numerical analysis carried out in Chapter IV.

Sustaining Arc

For a finite time interval of flight along a sustaining arc, we have

$$p_3 - wp_5 = 0 \tag{III.97}$$

The lift coefficient is given by (III.42) with A = 0. We have

$$\lambda^2 = 1 + \frac{2}{u^5}(u^4 - w^2) \tag{III.98}$$

On the other hand, from Eq. III.44, by putting A = 0, $p_4 = a_4$, we have the relation between the weight and the velocity along a sustaining arc using variable lift

$$\frac{u^3 w^2 (u+2)(u^4 - w^2)}{[u^4 - (u+1)w^2]^2} = a_4^2 \tag{III.99}$$

Explicitly, using $a = a_4^2$ as parameter, we have

$$w = u^2 \sqrt{\frac{[u^3(u+2) + 2a(u+1)] + u^2\sqrt{(u+2)[u^2(u+2)+4a]}}{2[u^3(u+2) + a(u+1)^2]}} \tag{III.100}$$

Fig. III. 8 plots the family of sustaining arc in the (w, u) space. The curves are bounded by two limiting curves.

For $a \to 0$, we have the parabola

$$w = u^2 \tag{III.100a}$$

For $a \to \infty$, we have the curve

$$w = \frac{u^2}{\sqrt{u+1}} \tag{III.100b}$$

Using (III.100) we can express the variable lift control in terms of u, and the constant a as

$$\lambda^2 = 1 + \frac{u^2(u+2) + 2a(u+1) - u\sqrt{(u+2)[u^2(u+2)+4a]}}{[u^3(u+2) + a(u+1)^2]} \tag{III.101}$$

For $u \to 0$, $\lambda \to \sqrt{3}$. For $u \to \infty$, $\lambda \to 1$. Hence, we see that, when variable lift is used along a sustaining arc

$$1 < \lambda < \sqrt{3} \qquad \text{(III.102)}$$

The family of lift curves is bounded by two limiting curves. When $a \to 0$, Eq. (III.101) becomes

$$\lambda = 1 \qquad \text{(III.101a)}$$

When $a \to \infty$, we have the limiting curve

$$\lambda = \sqrt{\frac{u+3}{u+1}} \qquad \text{(III.101b)}$$

For the bank angle, we have in general

$$\tan^2\sigma = \frac{1}{w^2}(\lambda^2 u^4 - w^2) \qquad \text{(III.103)}$$

Using (III.98) we have

$$\tan^2\sigma = \frac{(u+2)}{u}\left(\frac{u^4}{w^2} - 1\right) \qquad \text{(III.104)}$$

Another expression for $\tan\sigma$ can be obtained by noticing that Eq. (III.90) is also valid for sustaining flight, and that $\Lambda = w \tan\sigma$. Hence we can write the equation for $\tan\sigma$ with $a_4 \neq 0$

$$\tan^2\sigma - \frac{2u}{a_4}\tan\sigma - \left(1 + \frac{u^4}{w^2}\right) = 0 \qquad \text{(III.105)}$$

Combining the last two equations we have

$$\tan\sigma = \frac{a_4}{u^2}\left[\frac{u^4}{w^2} - (u+1)\right] \qquad \text{(III.106)}$$

For σ to have positive value, we notice that $a_4 < 0$, when $w > u^2/\sqrt{u+1}$. By eliminating w^2 between the two equations (III.104) and (III.106) we have the equation for $\tan\sigma$.

$$\tan^2\sigma - \frac{u(u+2)}{a_4}\tan\sigma - (u+2) = 0 \qquad \text{(III.107)}$$

Since $a_4 < 0$, we take the positive root

$$\tan\sigma = \frac{1}{2a_4}\left[u(u+2) - \sqrt{(u+2)[u^2(u+2) + 4a_4^2]}\right] \qquad \text{(III.108)}$$

Fig. III.9 plots the variation of the bank angle along a sustaining arc in terms of the nondimensional velocity, using a_4 as parameter.

When $u \to 0$, $\tan\sigma \to \sqrt{2}$ and $\sigma \to 54°44'$.

When $u \to \infty$ $\sigma \to 90°$ or $0°$ depending on the sign of a_4.

The family of curves is bounded by the limiting curve obtained when $-a_4 \to \infty$. We have

$$\tan\sigma = \sqrt{u+2} \tag{III.108a}$$

The variable thrust profile is obtained by taking the derivative of the Eq. (III. 99), using the appropriate relations for du/dt and dw/dt. After some algebraic manipulation, we have

$$\tau = \frac{2u[(u+1)D - N}{D} \times \frac{u^3[(2u+5)N - D] + a(3u+4)[(u+1)N - D]}{u^3[(2u^2+6u+5)N - (u+1)^2D] + a(2u^2+5u+4)[(u+1)N - D]} \tag{III.109}$$

where

$$N = [u^3(u+2) + 2a(u+1)] + u^2\sqrt{(u+2)[u^2(u+2)+4a]}$$
$$D = 2[u^3(u+2) + a(u+1)^2] \tag{III.110}$$

Fig. III. 10 plots the variation of the variable dimensionless thrust τ in terms of the dimensionless velocity u, using a as parameter. The family of curves is bounded by two limiting curves.

When $a \to 0$

$$\tau = \frac{4u^2}{u+2} \tag{III.109a}$$

When $a \to \infty$

$$\tau = \frac{2u^2(u+2)(3u+4)}{(u+1)(2u^2+5u+4)} \tag{III.109b}$$

It is seen that τ is increasing with u. Hence, since along a sustaining arc, the velocity is decreasing, the variable thrust is decreasing along that arc. Also we notice that along the sustaining arc, for small velocity the variable thrust magnitude remains at relatively low level.

Boosting Arc

For variable lift control, no analytical solution can be found for the boosting arc, but it is seen that in the (w, u) space the trajectory tends to the exponential curve (III. 48) when $\tau_M \to \infty$.

Optimal Trajectory

As for the case of rectilinear flight, the optimal trajectory is of the six types

BC		CB
BSC	and	CSB
BSB		CSC

The optimal trajectory is obtained by joining different types of subarcs such that the end conditions are satisfied.

To facilitate the computational program, we may use the following criteria (Fig. III.11).

Proposition 1

When maximum lift is used

$$\text{If} \qquad w_0 > \lambda_M u_0^2$$

The initial arc is a boosting arc.

$$\text{If} \qquad w_0 < \frac{\lambda_M u_0^2}{\sqrt{u_0} + 3}$$

the initial arc is a coasting arc.

When normal lift is used

$$\text{If} \qquad w_0 > u_0^2$$

the initial arc is a boosting arc.

$$\text{If} \qquad w_0 < \frac{u_0^2}{\sqrt{u_0} + 1}$$

the initial arc is a coasting arc.

Proposition 2

When maximum lift is used

$$\text{If} \qquad w_f > \lambda_M u_f^2$$

the final arc is a coasting arc.

$$\text{If} \qquad w_f < \frac{\lambda_M u_f^2}{\sqrt{u_f} + 3}$$

the final arc is a boosting arc.

When <u>normal lift is used</u>

If $$w_f > u_f^2$$

the final arc is a coasting arc.

If $$w_f < \frac{u_f^2}{\sqrt{u_f + 1}}$$

the final arc is a boosting arc.

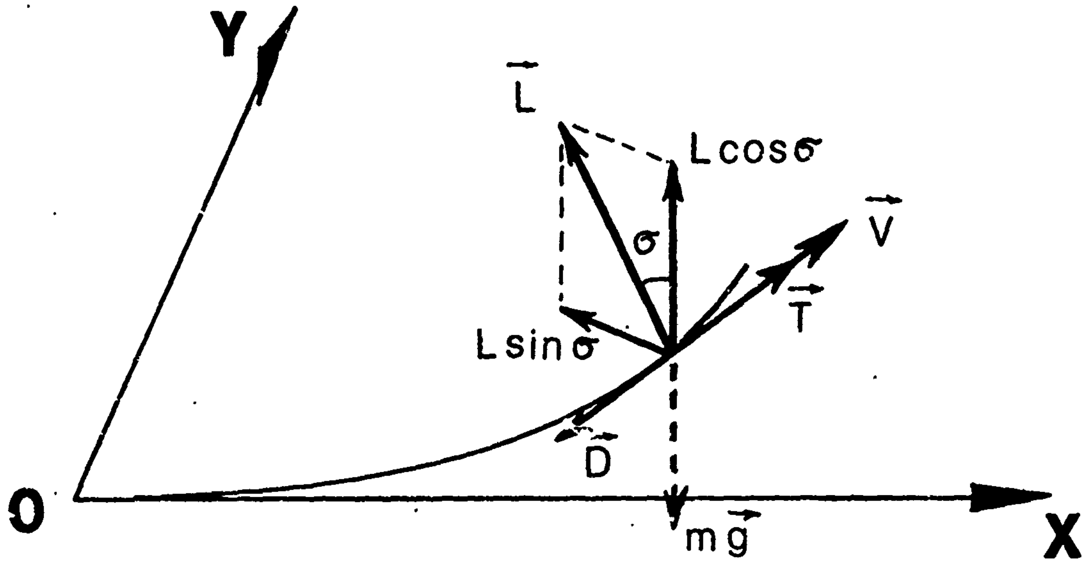


Fig. III. 1 Flight in a Horizontal Plane

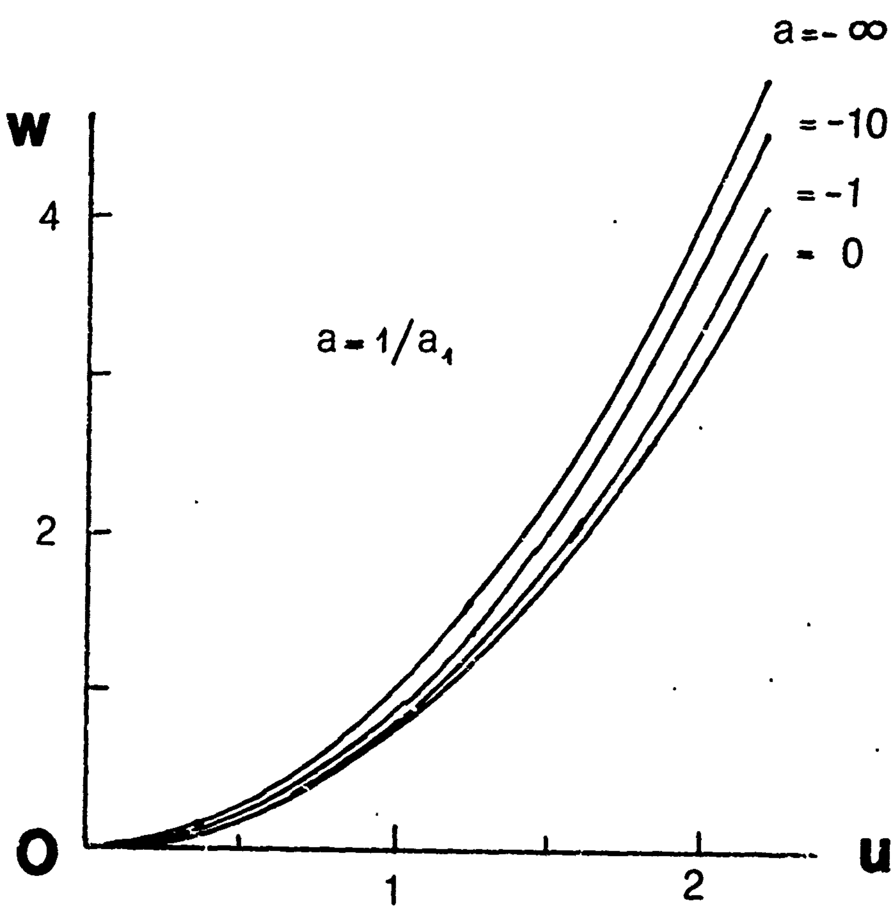


Fig. III. 2 Optimal Trajectories for Rectilinear Sustaining Flight

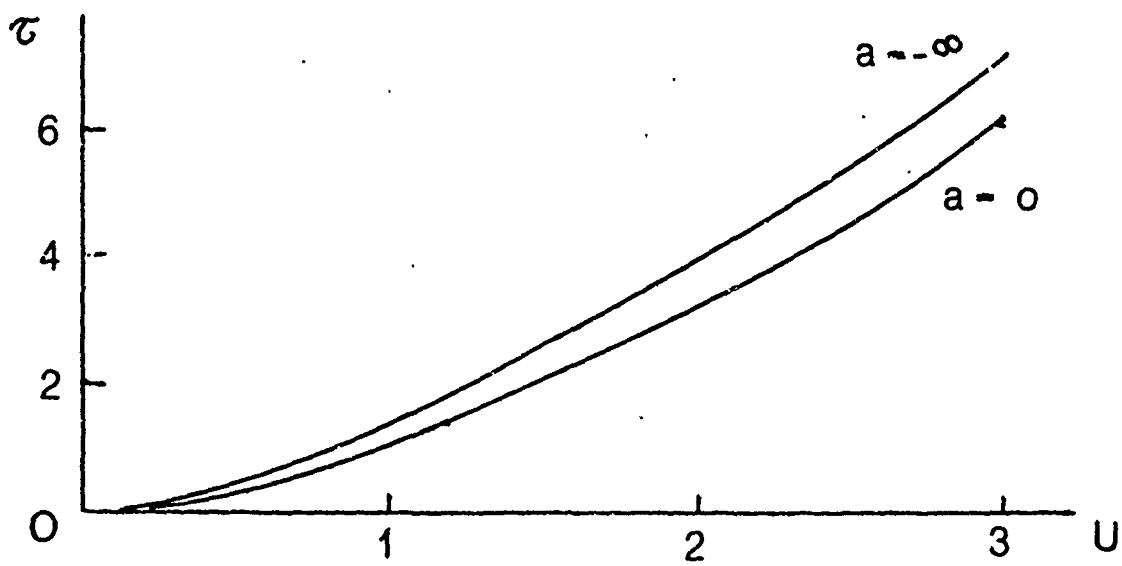


Fig. III. 3 Variable Thrust Profile for Rectilinear Sustaining Flight

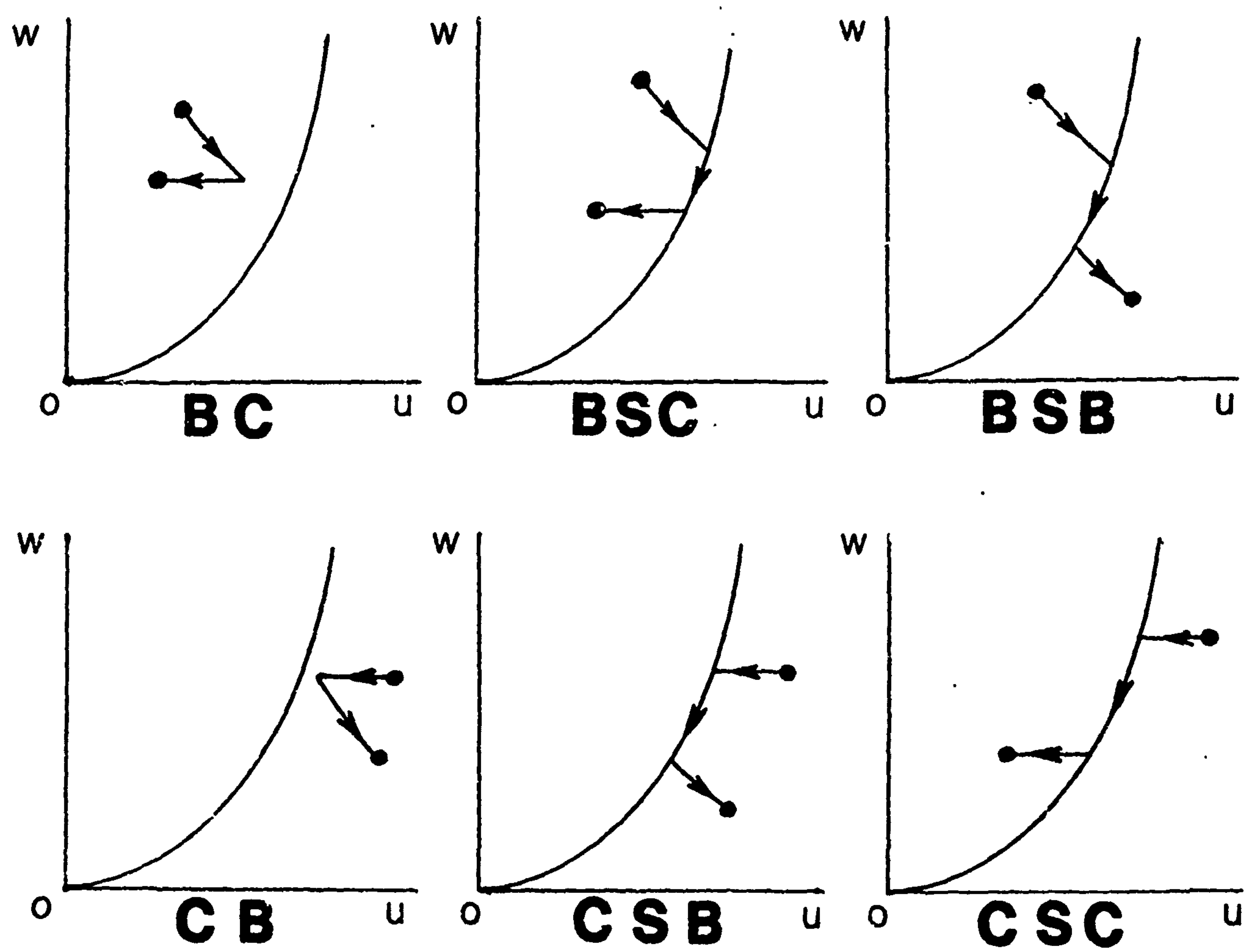


Fig. III. 4 Optimal Trajectories for Rectilinear Flight

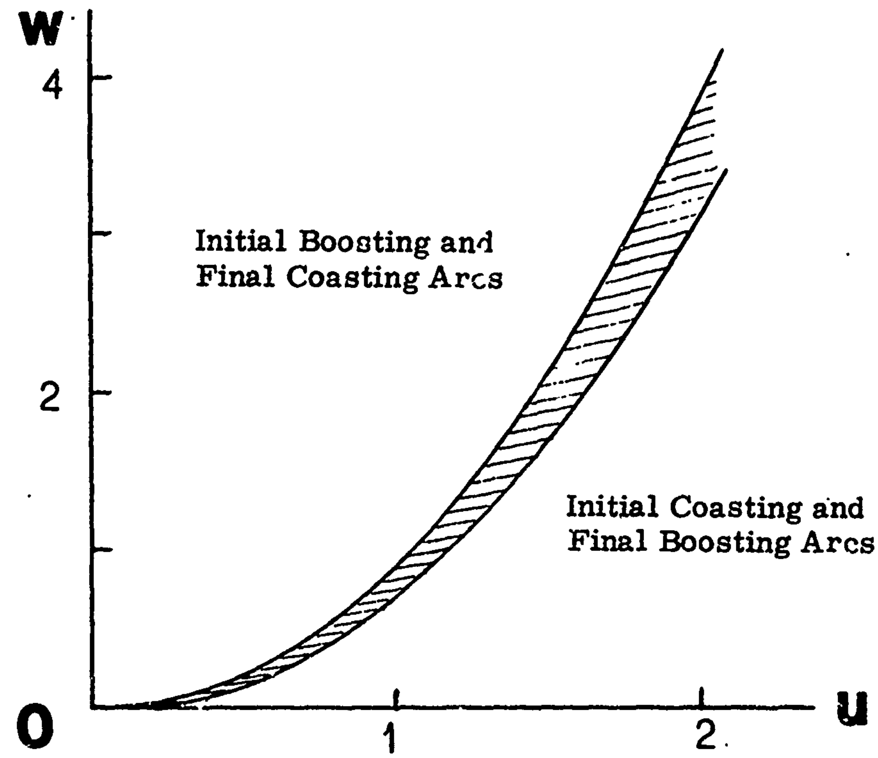


Fig. III. 5 Criteria for the Selection of the Initial and Final Arcs

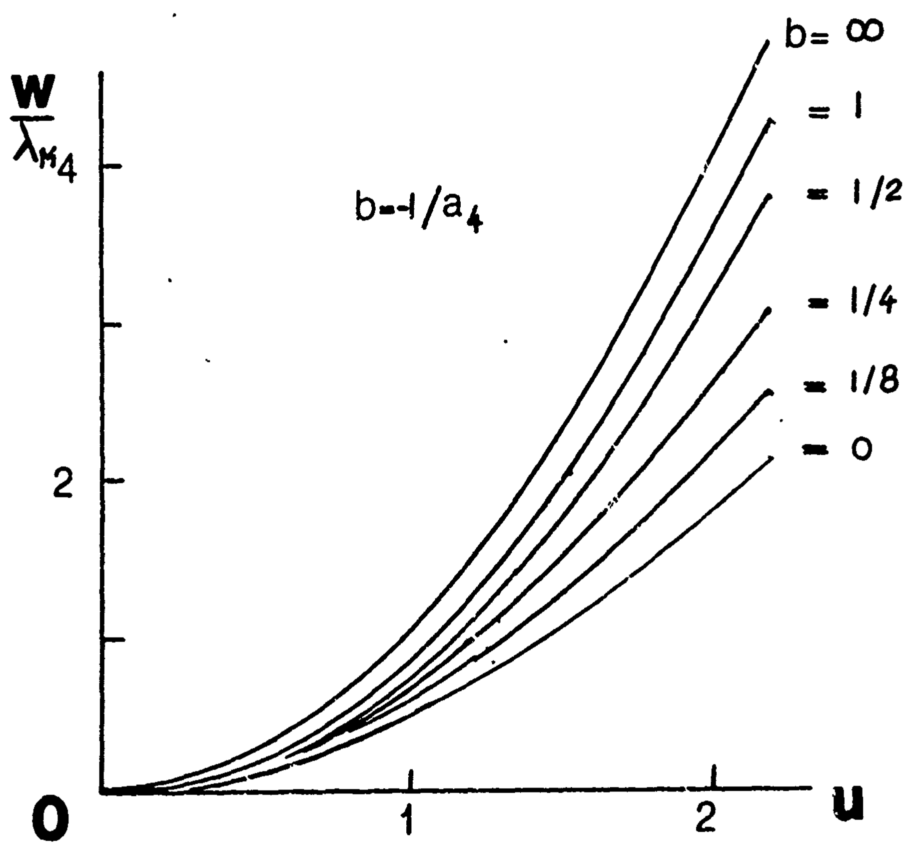


Fig. III. 6 Optimal Turn Trajectories for Sustaining Flight at Maximum Lift

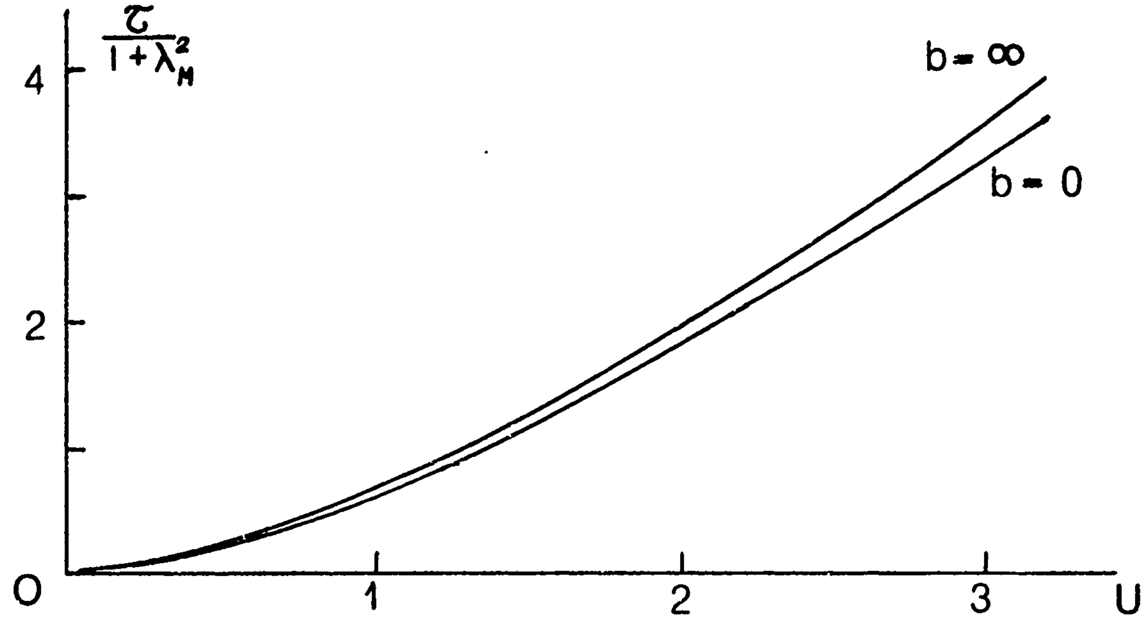


Fig. III. 7 Variable Thrust Profile Along Sustaining Arc for Turning at Maximum Lift

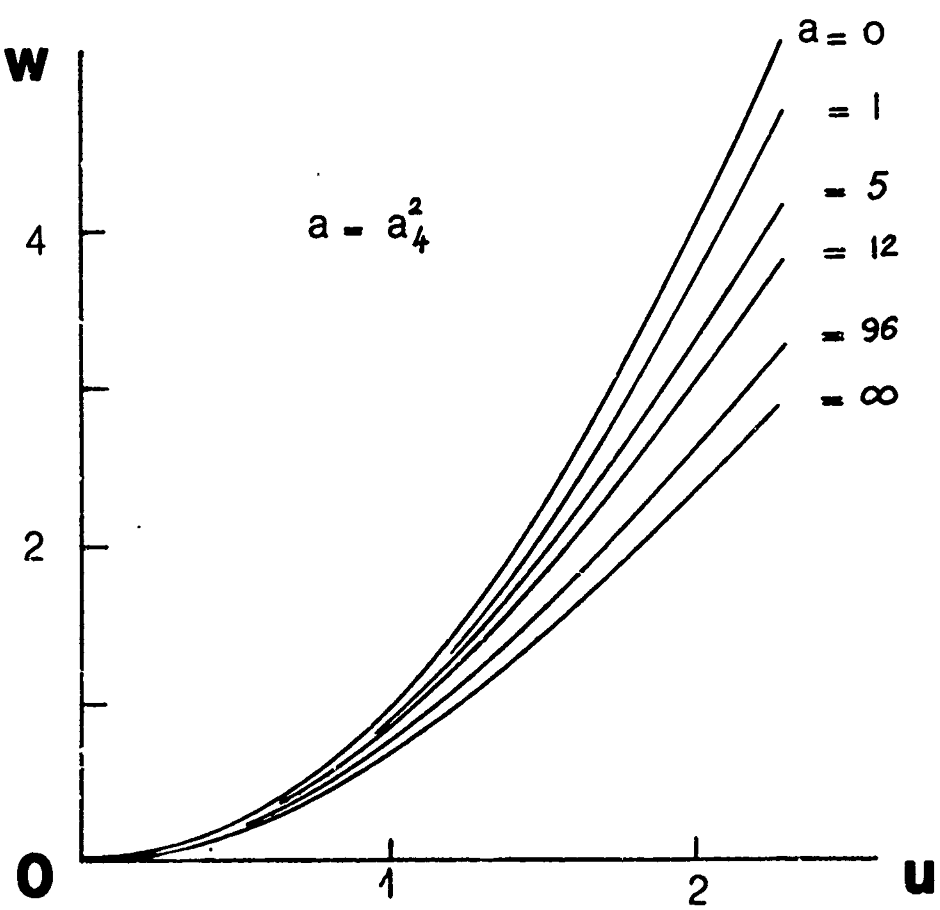


Fig. III. 8 Optimal Turn Trajectories for Sustaining Flight with Variable Lift

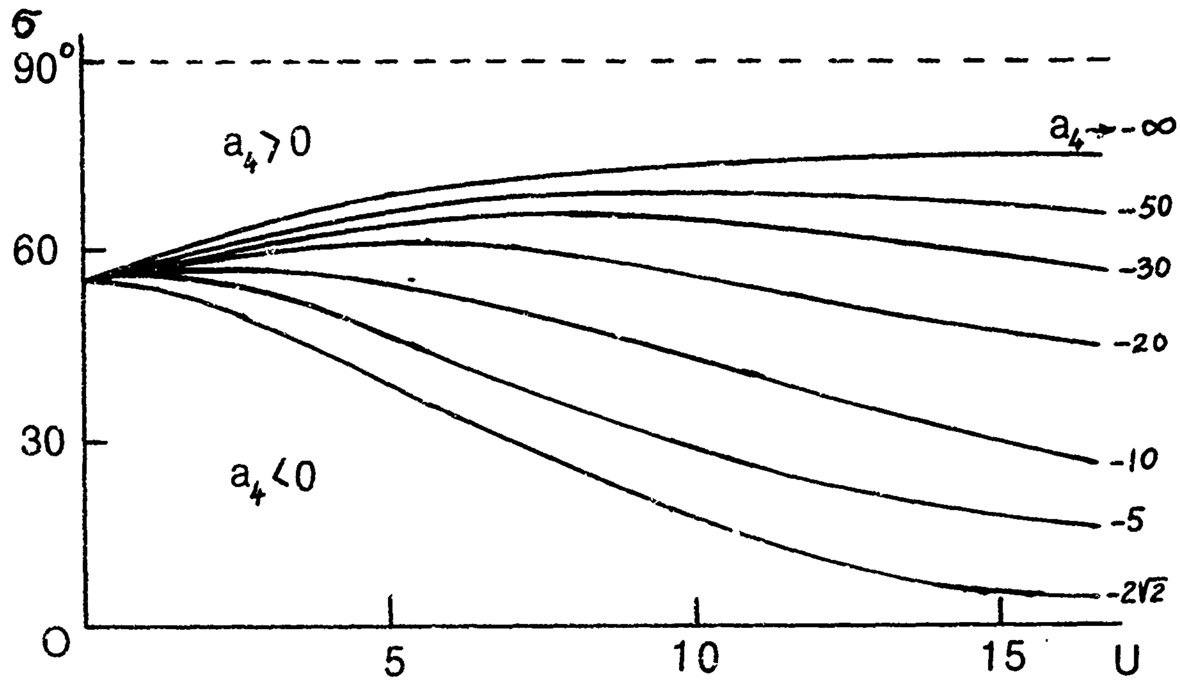


Fig. III. 9 Optimal Variable Bank Angle for Turning Flight Along Sustaining Arc

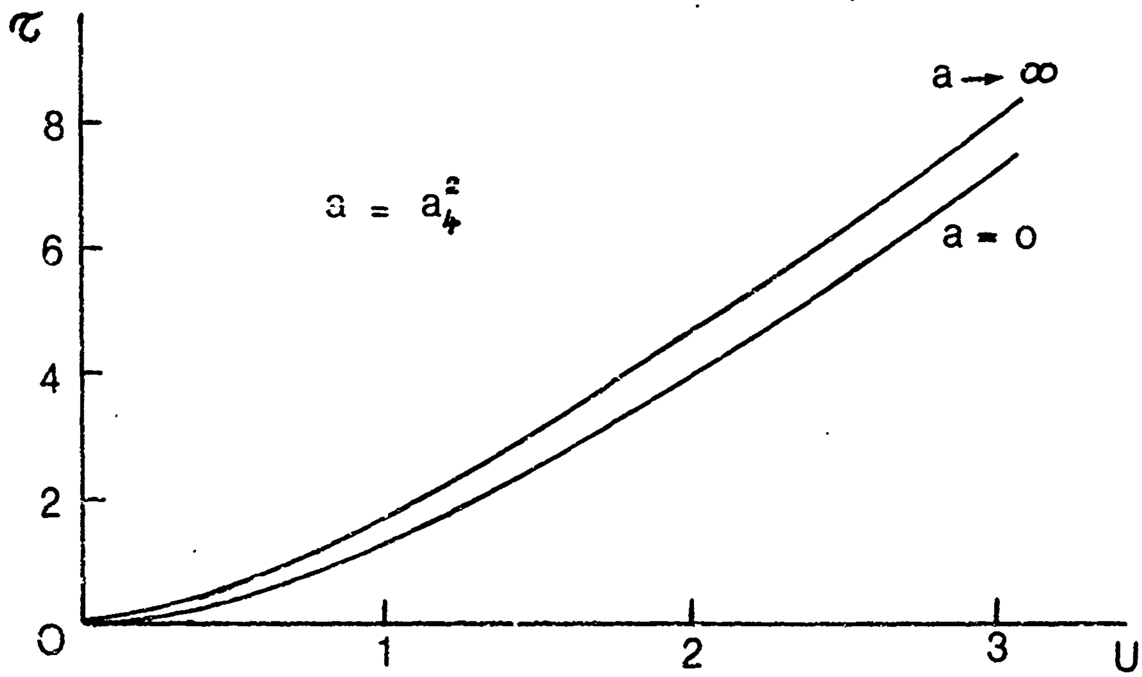
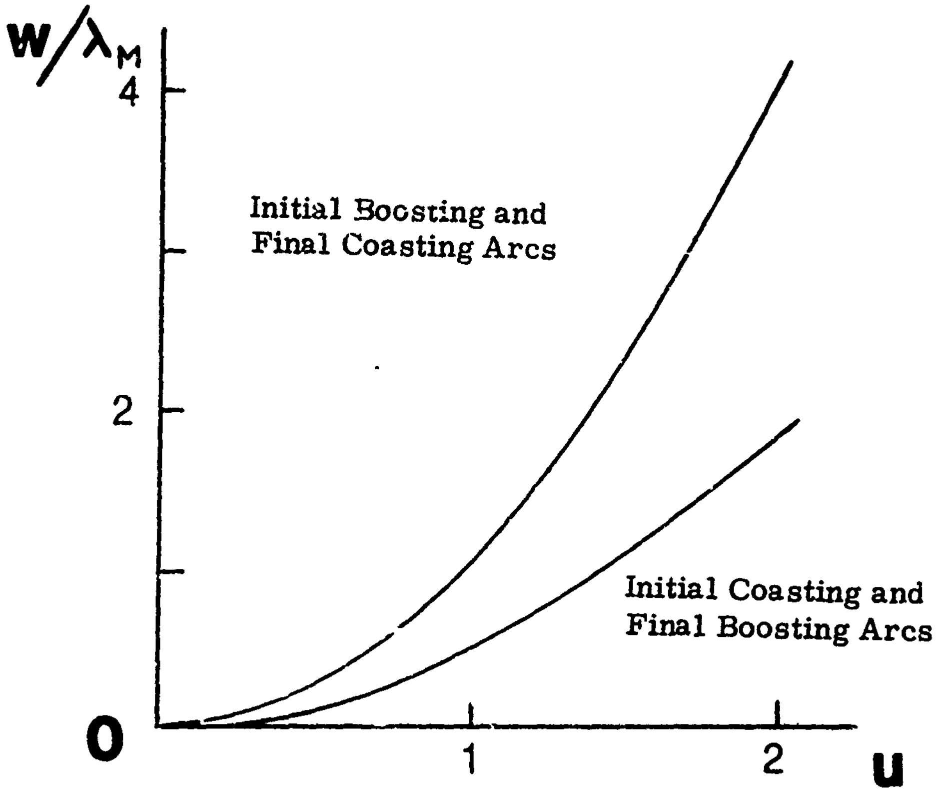
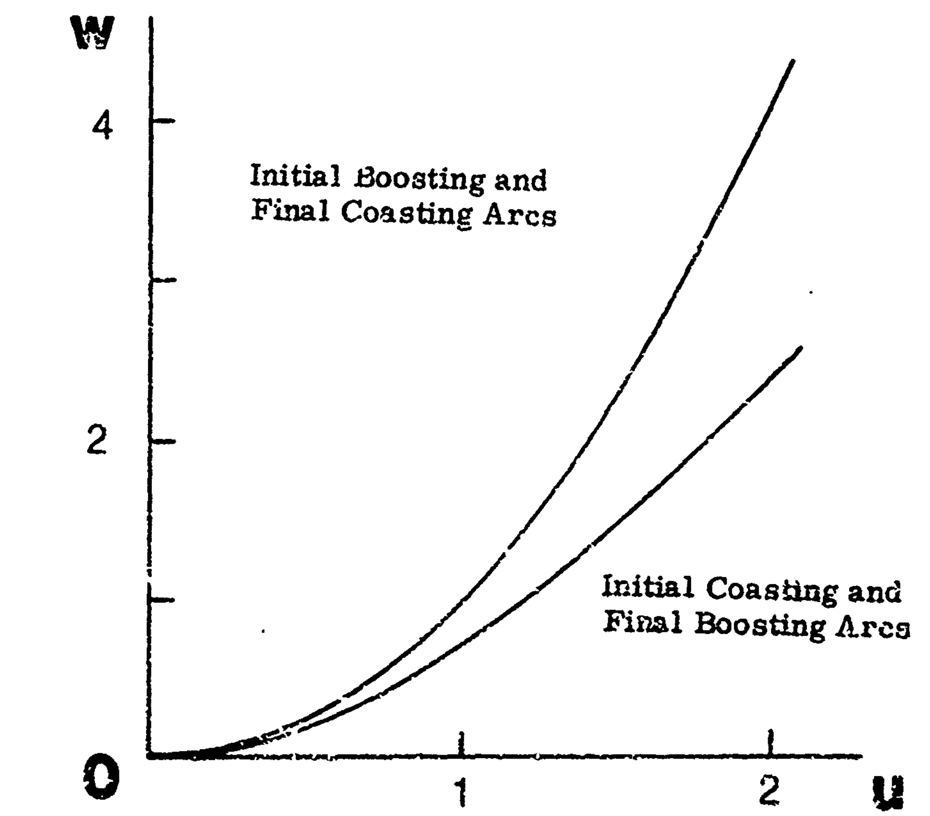


Fig. III. 10 Variable Thrust Profile for Turning Flight Along a Sustaining Arc Using Variable Lift



Maximum Lift



Normal Lift

Fig. III. 11 Criteria for the Selection of the Initial and Final Arcs

IV. NUMERICAL RESULTS

In this chapter a discussion of the computer program used to compute solutions to fully constrained problems will be presented along with simulation results for representative cases.

IV.1. Basic Optimization Problem

The equations of motion for flight in a horizontal plane are given by Eqs. (III.1) and (III.2). To facilitate the discussion of this section, the state variables (i.e., x, y, v, β, m) will be denoted by $x_1, \ldots, x_n$, where $n = 5$ for this problem, and the controls τ, σ, C_L will be denoted by u_1, u_2, u_3, respectively. The computer program is developed in the (x, u)-system.

The problems considered in this study are Mayer type optimal control problems, and the computer program was developed for the following Mayer problem.

MINIMIZE: $\tilde{J} = \phi(t_f, x_f)$ (IV.1)

SUBJECT TO: $\dot{x} = \tilde{f}(t, x; u_1, u_2, u_3)$ (IV.2)

$x(t_0) = x_0$ (IV.3)

$0 \leqq u_1 \leqq T_{max}$ (IV.4)

$u_3 \leqq C_{L_{max}}$ (IV.5)

$G(x, u_2, u_3) = 0$ (IV.6)

Equation (IV.1) is the performance index which contains the quantity to be optimized (e.g., t_f for minimum time problems) and penalty terms for the terminal boundary conditions. Equations (IV.2), (IV.4) and (IV.5), and (IV.6) correspond to Equations (III.1), (III.3), and (III.2), respectively. The control equality constraint (IV.6) can be used to eliminate either u_2 or u_3 from the problem. Since the main purpose of the program is to verify and extend the results of Chapters II and III, C_L is eliminated, i.e.,

$$G(x, u_2, u_3) = 0 \Rightarrow u_3 = g(x, u_2), \qquad \text{(IV.7)}$$

or, in physical variables·

$$L \cos\sigma = mg \quad \Rightarrow \quad C_L = \frac{2mg}{\rho V^2 S \cos\sigma} \tag{IV.8}$$

We now have the following problem: Minimize Eq. (IV.1) subject to $x(t_0) = x_0$ and:

$$\dot{x} = f(t, x, u_1, u_2) \equiv \tilde{f}[t, x, u_1, u_2, g(x, u_2)] \tag{IV.9}$$

$$0 \leqq u_1 \leqq T_{max} \tag{IV.10}$$

$$g(x, u_2) \leqq C_{L_{max}} \,. \tag{IV.11}$$

IV.2. Solution by Gradient-Type Methods

The first step in the development of a function space gradient-type method for optimal control problems is the definition of the following augmented functional:

$$J = \phi(t_f, x_f) + \int_{t_0}^{t_f} \lambda^T(t)[f(t, x, u_1, u_2) - \dot{x}]dt \tag{IV.12}$$

where $\lambda(t)$ is a vector of influence functions which in the limit (of the numerical algorithm) approaches the $p(t)$-vector of Chapter III.

It is convenient to define a Hamiltonian function

$$H(t, x, \lambda, u) \equiv \lambda^T f(t, x, u). \tag{IV.13}$$

Let $u^{(0)}(t) \equiv (u_1^{(0)}(t), u_2^{(0)}(t))$ be an estimated control vector which satisfies Inequalities (IV.10) and (IV.11), and let $t_f^{(0)}$ be an estimate of the final time. One can then integrate Eqs. (IV.9) forward to define a corresponding trajectory $x^{(0)}(t)$, defined on $[t_0, t_f^{(0)}]$, and a value for the performance index, $J[u^{(0)}]$.

Let $\lambda^{(0)}(t)$ be an arbitrary continuous vector which we shall characterize later. To develop the gradient-class of numerical algorithms, expand Eq. (IV.12) to first-order about the pair $(x^{(0)}(t), u^{(0)}(t))$ with the following definitions

$$u^{(1)}(t) = u^{(0)}(t) + \delta u(t) \tag{IV.14}$$

$$x^{(1)}(t) = x^{(0)}(t) + \delta x(t) \tag{IV.15}$$

$$t_f^{(1)} = t_f^{(0)} + dt_f. \qquad \text{(IV.16)}$$

$$x^{(1)}(t_f^{(1)}) = x^{(0)}(t_f^{(0)}) + dx_f \qquad \text{(IV.17)}$$

Then,

$$J[u^{(1)}] = J[u^{(0)}] + \phi_{t_f}^{(0)} dt_f + \phi_{x_f}^T dx_f$$
$$+ [H(t_f^{(0)}) - \lambda^{(0)^T}(t_f^{(0)})\dot{x}^{(0)}(t_f^{(0)})]dt_f$$
$$+ \int_{t_0}^{t_f^{(0)}} [H_x^{(0)^T} \delta x + H_u^{(0)^T} \delta u - \lambda^{(0)^T} \delta\dot{x}]dt . \qquad \text{(IV.18)}$$

Upon integration by parts of the third term in the integrand and rearrangement of terms, the following equation expresses the difference in cost between the base trajectory and the first iterate (to first-order):

$$\Delta J[\delta u] = J[u^{(1)}] - J[u^{(0)}] = (\phi_{t_f}^{(0)} + H^{(0)})_{t_f^{(0)}} dt_f + (\phi_{x_f}^{(0)} - \lambda^{(0)})^T_{t_f^{(0)}} dx_f$$
$$+ \int_{t_0}^{t_f^{(0)}} [(H_x^{(0)} + \dot{\lambda}^{(0)})^T \delta x + H_u^{(0)T} \delta u]dt , \qquad \text{(IV.19)}$$

where

$$\delta x(t_f^{(0)}) = dx_f - \dot{x}^{(0)}(t_f^{(0)})dt_f \qquad \text{(IV.20)}$$

has been used to eliminate $\delta x(t_f^{(0)})$ from Eq. (IV.19).

We now characterize $\lambda^{(0)}(t)$ so that a stable iterative algorithm (i.e., $\Delta J \leqq 0$) is defined.

SPECIFY: $\lambda^{(0)}(t_f^{(0)}) = \phi_{x_f}^{(0)}$ (IV.21)

$$\dot{\lambda}^{(0)}(t) = -H_x[t, x^{(0)}(t), u^{(0)}(t), \lambda^{(0)}(t)] . \qquad \text{(IV.22)}$$

Then,

$$\Delta J = (\phi_{t_f}^{(0)} + H^{(0)})_{t_f^{(0)}} dt_f + \int_{t_0}^{t_f^{(0)}} H_u^{(0)^T} \delta u \, dt, \qquad \text{(IV.23)}$$

or,

$$\Delta J = \int_{t_0}^{t_f^{(0)}} \{[(\phi_{t_f}^{(0)} + H^{(0)})/(t_f^{(0)} - t_0)]dt_f + H_u^{(0)^T} \delta u\} dt. \tag{IV.24}$$

Note that if the following choices are made:

$$dt_f = t_f^{(1)} - t_f^{(0)} = -\epsilon[(\phi_{t_f}^{(0)} + H^{(0)})/(t_f^{(0)} - t_0)] \tag{IV.25}$$

$$\delta u(t) = u^{(1)}(t) - u^{(0)}(t) = -\epsilon H_u^{(0)}(t) \text{ (subject to constraints)} \tag{IV.26}$$

with $\epsilon > 0$ (small), then $\Delta J \leqq 0$. These choices represent the gradient method choices.

In the computer program developed here, the conjugate gradient method [Refs 7,8] was employed. It can be shown that this method also guarantees $\Delta J \leqq 0$. We shall now present the algorithm in two parts; first, the algorithm with no control inequality constraints will be listed, and second, the modifications necessary to handle the control inequality constraints will be listed

UNCONSTRAINED CONJUGATE GRADIENT ALGORITHM

1.) Guess $u^{(0)}(t)$, $t_f^{(0)}$.

2.) Compute: $x^{(I)}(t)$, $\lambda^{(I)}(t)$, $H_u^{(I)}(t)$

$$p^{(I)}(t) = H_u^{(I)}(t) + \frac{\int_{t_0}^{t_f^{(I)}} H_u^{(I)^T} H_u^{(I)} dt}{\int_{t_0}^{t_f^{(I-1)}} H_u^{(I-1)^T} H_u^{(I-1)} dt} p^{(I-1)}(t) \tag{IV.27}$$

$$p^{(0)}(t) \equiv H_u^{(0)}(t)$$

3.) Check $\int_{t_0}^{t_f^{(I)}} H_u^{(I)T} H_u^{(I)} dt \leqq \epsilon$. If yes, stop. If no, go to 4).

4.) Perform one-dimensional (1-D) search to determine the value α_I which minimizes

$$J[u^{(I)} - \alpha_I p^{(I)}]. \tag{IV.28}$$

5.) Define $u^{(I+1)} = u^{(I)} - \alpha_I p^{(I)}$, set $I = I + 1$. Return to 2.)

Equation (IV.27) defines the conjugate gradient search direction. Discussions of this method are given in Refs [7,8].

MODIFICATION FOR $0 \leqq u_1 \leqq T_{max}$

Let $W_1^{(I)}$ be the set of points at which $u_1^{(I)}(t)$ is on the boundary, i.e.,

$$W_1^{(I)} \equiv \{t | u_1^{(I)}(t) = 0 \text{ or } T_{max}\} . \tag{IV.29}$$

Replace the u_1-part of the inner products in Eq. (IV.27) by

$$\int_{[t_0, t_f^{(I)}] - W_1^{(I)}} H_{u_1}^{(I)} H_{u_1}^{(I)} dt, \tag{IV.30}$$

i.e., in the evaluation of the inner product associated with $H_{u_1}^{(I)}(t)$ do not include boundary subarcs.

The set $W_1^{(I+1)}$ is formed during the 1-D search. The implementation is as follows. Let $\tilde{\alpha}$ be a candidate search parameter:

$$\begin{aligned} &\text{if } u^{(I)}(t) - \tilde{\alpha} p^{(I)} > T_{max}, \text{set } \tilde{u}^{(I+1)}(t) = T_{max} \\ &\text{if } u^{(I)}(t) - \tilde{\alpha} p^{(I)} < 0 \quad\quad, \text{set } \tilde{u}^{(I+1)}(t) = 0. \end{aligned} \tag{IV.31}$$

After the function evaluations for the 1-D search are computed, a value α_I is obtained by a cubic interpolation process. The set $W_1^{(I+1)}$ is then defined as the set of all t which cause the inequalities in Eq. (IV.31) with α_J replacing $\tilde{\alpha}$.

MODIFICATION FOR $g(x, u_2) \leqq C_{L_{max}}$

As noted in Chapter III the constraint on bank angle is a function of state variables. Thus, this control constraint is treated in a manner slightly different than the thrust, u_1. Let $W_2^{(I)}$ be the set of points at which $g(x^{(I)}(t), u_2^{(I)}(t)) = C_{L_{max}}$. In the computer program the control vector is stored at each integration step, and linear interpolation is used to define the control between the fixed step-lengths. As with $W_1^{(I+1)}$, $W_2^{(I+1)}$ is determined in the 1-D search. However, in the forward integration for the function evaluation for a specified search parameter, γ, the control constraint relation is changing because x appears in $g(x,u_2) = C_{L_{max}}$. Thus, when the integrator is at t_i, the constraint at $t_i + \Delta t$ is predicted by

$$g[x(t_i)+\dot{x}(t_i)\,\Delta t,\ u_2\,(t_i + \Delta t)] \leqq C_{L_{max}}\,. \tag{IV.32}$$

If the integration stepsize is sufficiently small, this approximation is sufficient to approximate the control constraint at $t_i + \Delta t$. Except for this modification, the set $W_2^{(I+1)}$ is determined in the same way that $W_1^{(I+1)}$ is determined.

IV.3. Deck Description

The program is developed based on the conjugate-gradient algorithm. A Runge-Kutta fourth order integration scheme is used to perform both forward and backward integration. A switch is imposed such that it can be set to the gradient algorithm.

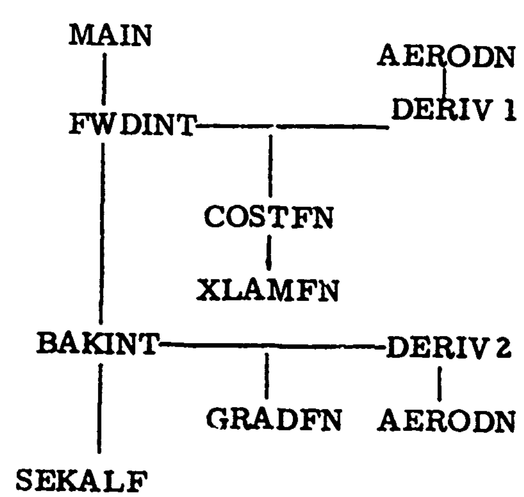
The program is designed to minimize a weighted performance index which includes the following effects.

1. Total flight time t_f.
2. Terminal states x_f, y_f, v_f, β_f and m_f.

The performance index is

$$J = c\,t_f + p_1(x - x_f)^2 + p_2(y - y_f)^2 + p_3(v - v_f)^2 + p_4(\beta - \beta_f)^2 + p_5(m - m_f)^2$$

9 subroutines are used to handle this problem:



MAIN: Reads in all necessary data, initial and terminal values, controls the application of the conjugate gradient algorithm, calls the forward and backward integration routines, directs the one dimensional search, updates the control vector and terminal time and prints out a message concerning the results of the iteration and prints out the control profiles obtained by that iteration.

A. J = An integer which indicates where the data is located.

B. Namelist input data

G = gravitational constant

RHO = atmospheric density

CDO = zero-lift drag coefficient (C_{D_0})

CE = exhaust velocity

DK = induced-drag coefficient (k)

AREA = aerodynamic reference area

DELTS = integration stepsize

CLA = slope of the lift curve (C_{L_α})

ITMAX = limit on number of conjugate gradient iterations

ITMX = limit on steps in 1-D search

KOUNTM = limit on iterations for weight cutoff

IKEY = number of iterations to reset the search direction

CSTR = estimate of final cost, initial guess of parameter α for one-dimensional search

PFUN(5) = penalty coefficient vector

CCOST(1) = coefficient in cost functional

B = not used

I = Index for output device

JERK = switch for scheme option = 1 conjugate gradient method
= 2 gradient method

DTFM = maximum allowable final time change

XDTFM = fraction of DTFM used to start 1-D search

XO = initial position in x-direction

YO = initial position in y-direction

UO = initial velocity

BO = initial heading

WO = initial mass

TO = initial time

XF = final position in x-direction

YF = final position in y-direction

UF = final velocity

BF = final heading

WF = final mass

IOUT = print frequency for forward integration

IOUT 2 = print frequency for backward integration

IPRNT 1 = 1 for initial run, = 2 for continuing run after normal termination of conditional cutoff

IPRNT 2 = not used

CBND(2,2) = control bound

ERRMX = error tolerance for integration routine

ERRMN = not used

TCUT = upper time limit on trajectory

EPST = cutoff tolerance for norm of control change

EPSTF = not used

EPSA = cutoff tolerance for integration weight cutoff

EPSIT = cutoff tolerance on gradient norm

ERR = cutoff tolerance for small cost change

Note: for any final state unspecified an arbitrary number other than zero may be assigned.

C. Control Vector Date

IJKU = total number of control points

TF = initial guessed t_f

U(3,IJKU) = control vector and time point

D. Bounded Thrust Value

NN = number of points with bounded thrust

TAUMAX(N) = maximum thrust limit at Nth point

TAUMIN(N) = minimum thrust limit at Nth point

SEKALF: (One-dimensional search subroutine): Determines the parameter for the new control value in the conjugate gradient algorithm. Fits a cubic in α to known values of $J(\alpha)$, $\partial J/\partial\alpha$, to obtain min $J(\alpha)$ and then α^* for J min.

FWDINT: Subroutine performs the forward integration of the state variables and calls the subroutines to evaluate the cost functional and final multiplier values.

BAKINT: Subroutine performs the backward integration of the state variables and multiplier equations, calls on GRADFN to calculate the gradient and store the value at each integration step, determines the new search direction.

DERIV1: Subroutine calculates the time derivatives of the state variables
DERIV2: Subroutine calculates the time derivatives of the multipliers
AERODN: Calculates aerodynamic parameters
XLAMFN: Calculates final multiplier values
COSTFN: Calculates cost functional
GRADFN: Calculates gradient values at each step.

IV.4. Representative Problems and Numerical Solutions

Several minimum-time missions have been selected and solved by the numerical programs. The missile is assumed to have a parabolic drag polar

$$C_D = C_{D_0}(M) + k(M)C_L^2 \qquad (IV.33)$$

In terms of the angle of attack, the lift coefficient is given by

$$C_L = C_{L_\alpha}(M)\alpha \qquad (IV.34)$$

where C_{L_α} is the slope of the lift curve

$$C_{L_\alpha} = \left(\frac{\partial C_L(M,\alpha)}{\partial\alpha}\right)_{\alpha=0} \qquad (IV.35)$$

The induced drag coefficient $k(M)$ can be related to the aerodynamic stability derivatives by observing that $C_D(M,\alpha)$ can be expanded in Taylor's series as

$$C_D = C_{D_0}(M) + \left(\frac{\partial C_D}{\partial\alpha}\right)_{\alpha=0}\alpha + \frac{1}{2}\left(\frac{\partial^2 C_D}{\partial\alpha^2}\right)_{\alpha=0}\alpha^2 + \cdots \qquad (IV.36)$$

For a parabolic representation, as given by (IV.33), the coefficient of the second term on the right hand side of Eq. (IV.36) is negligible and we have

$$k(M)C_L^2 = k(M)C_{L_\alpha}^2(M)\alpha^2 = \frac{1}{2}\left(\frac{\partial^2 C_D}{\partial \alpha^2}\right)_{\alpha=0}\alpha^2$$

Hence

$$k(M) = \frac{C_{D_{\alpha\alpha}}(M)}{2C_{L_\alpha}^2(M)} \tag{IV.37}$$

where

$$C_{D_{\alpha\alpha}}(M) = \left(\frac{\partial^2 C_D}{\partial \alpha^2}\right)_{\alpha=0} \tag{IV.38}$$

It is customary to write

$$\frac{1}{2}C_{D_{\alpha\alpha}}(M) = \epsilon(M)\, C_{L_\alpha}(M) \tag{IV.39}$$

where $\epsilon(M)$ is the aerodynamic efficiency with typical values bounded by

$$\frac{1}{2} \leqq \epsilon(M) \leqq 1 \tag{IV.40}$$

Hence

$$k(M) = \frac{\epsilon(M)}{C_{L_\alpha}(M)} \tag{IV.41}$$

For a given Mach number, the maximum lift-to-drag ratio is given by

$$E^* = \frac{1}{2\sqrt{kC_{D_0}}} = \frac{1}{2}\sqrt{\frac{C_{L_\alpha}(M)}{\epsilon(M)C_{D_0}(M)}} \tag{IV.42}$$

To verify the analytical solutions obtained in Chapter III, in the first part of this numerical analysis, Part A, we shall assume that the coefficients C_{D_0}, C_{L_α}, and k are constant. We shall indicate in Part B the necessary modification in the subprograms when these aerodynamics derivatives vary as functions of the Mach number.

A. Constant Aerodynamic Derivatives

The following values are used in the numerical computations.

Initial weight, W_0 = 861 lbs

Final weight, W_f = 434 lbs

Maximum thrust, T_M = 34,000 lbs

Minimum thrust, T_{min} = 0 lbs

Exhaust velocity, c = 8,050 ft/sec

Reference area, S = 0.66 ft^2

Air mass density at at 40,000 ft, ρ = 0.000585 slugs/ft^3

Zero lift drag coefficient, C_{D_0} = 0.3

Lift curve slope, C_{L_α} = 10.3

Induced drag coefficient, k = 0.097

Maximum angle of attack, α_M = 30°

Problem 1: Pure Coasting Flight

Initial Conditions	Terminal Conditions
$X_0 = 0$	X_f = free
$Y_0 = 0$	Y_f = free
$\beta_0 = 0$	$\beta_f = 45°$
V_0 = 2136.2 ft/sec ~ 2.2 Mach	$V_f \geqq$ 1000 ft/sec
W_0 = 861 lbs	W_f = 861 lbs

This case was considered to test of the efficiency of the program. After 12 iterations we obtain $t_{f_{min}}$ = 10.904 seconds with the final velocity being $V(t_f)$ = 1525 ft/sec. As predicted by the analytical solution, the flight is effected at maximum angle-of-attack, and the maximum allowable bank angle decreases monotonically along the optimal trajectory.

Problem 2: Minimum time turning with high initial velocity

Initial Conditions	Terminal Conditions
$X_0 = 0$	X_f = free
$Y_0 = 0$	Y_f = free
$\beta_0 = 0$	$\beta_f = 135°$

$V_0 = 2136.2$ ft/sec	$V_f \geqq 1000$ ft/sec
$W_0 = 861$ lbs	$W_f = 434$ lbs

In this problem, a thrusting phase is involved. Since the constraining final velocity is low the thrust profile is of the Boost-Coast type. The minimum time obtained is $t_f = 9.49$ seconds. The trajectory is flown with maximum angle-of-attack.

Keeping the same final heading, if we increase the final constraining velocity, there exists a critical final velocity such that the thrust profile reverses to the Coast-Sustain-Boost type.

Problem 3: Minimum time turning with low initial velocity

Initial Conditions	Terminal Conditions
$X_0 = 0$	X_f = free
$Y_0 = 0$	Y_f = free
$\beta_0 = 0$	$\beta_f = 45°$
$V_0 = 1000$ ft/sec	$V_f \geqq 1000$ ft/sec
$W_0 = 861$ lbs	$W_f = 434$ lbs

After 40 iterations, we obtain $t_{f_{min}} = 8.133$ seconds with the final velocity being $V(t_f) = 5700$ ft/sec. The optimal thrust profile is Boost-Coast with the trajectory flown at maximum angle-of-attack.

Problem 4: Minimum time turning to a specified terminal position

Initial Conditions	Terminal Conditions
$X_0 = 0$	$X_f = 31680$ ft = 6 miles
$Y_0 = 0$	$Y_f = 31680$ ft = 6 miles
$\beta_0 = 0$	β_f = free
$V_0 = 2136.2$ ft/sec	$V_f \geqq 1000$ ft/sec
$W_0 = 861$ lbs	$W_f = 434$ lbs

This problem is designed to force the appearance of a sustaining phase where variable thrust control is used. After 23 iterations we obtain $t_{f_{min}} = 27.1559$ seconds. The trajectory is composed of an initial coasting arc of 19 seconds, followed by a sustaining arc of 5 seconds with a final boosting arc of 3 seconds. The thrust level for the sustaining arc is very low and is nearly constant.

Fig. IV. 1 shows the convergence of the final coordinates X_f and Y_f, while Fig. IV. 2 shows the convergence of the final weight W_f.

Fig. IV. 3 shows the optimal thrust profile and Fig. IV. 4 presents the variation in the optimal bank angle.

<u>Problem 5:</u> Minimum time turning to a specified terminal position

Initial Conditions	Terminal Conditions
$X_0 = 0$	$X_f = 0$
$Y_0 = 0$	Y_f = 105600 ft = 20 miles
$\beta_0 = 0$	β_f = free
V_0 = 2136.2 ft/sec	$V_f \geqq$ 1000 ft/sec
W_0 = 861 lbs	W_f = 434 lbs

This problem is desinged to have a longer flying time, thus making the variable thrust control more prominent. After 35 iterations we obtain $t_{f_{min}}$ = 45.029 seconds. The trajectory is of the type Coast-Sustain-Coast.

Fig. IV. 5 shows the final trajectory, with the initial guessed trajectory. The figure illustrates the efficiency of the numerical program.

Fig. IV. 6 shows the optimal thrust profile and Fig. IV. 7 presents the variation of the optimal bank angle.

B. Mach Dependent Aerodynamic Parameters

The program has been assembled to include the case where the aerodynamic derivatives C_{D_0} (M) and k(M) in Eq. (IV.33) are functions of the Mach number. The following numerical data are available for the rocket considered.

Drag Coefficient $C_D(\alpha, M)$

α (deg)	Mach Number								
	0.0	0.8	1.0	1.05	1.5	2.0	2.2	3.5	6.0
0	0.25	0.20	0.30	0.45	0.40	0.33	0.30	0.22	0.15
5	0.30	0.26	0.40	0.50	0.45	0.38	0.34	0.27	0.18
10	0.51	0.48	0.70	0.80	0.75	0.61	0.60	0.48	0.41
15	1.06	1.07	1.30	1.41	1.35	1.18	1.14	1.00	0.92
20	2.10	2.22	2.60	2.67	2.60	2.20	2.10	1.90	1.67
25	3.87	4.25	4.60	4.68	4.30	3.48	3.30	3.05	2.84
30	-	4.90	5.00	-	5.60	-	4.60	-	-

Lift Coefficient $C_L(\alpha, M)$

α (deg.)	Mach Number							
	0.0	0.8	1.0	1.5	2.2	2.5	3.5	6.0
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.67	0.62	0.70	0.77	0.80	0.81	0.78	0.67
10	1.60	1.54	1.80	1.85	1.90	1.91	1.94	1.86
15	2.85	2.61	2.90	3.18	3.50	3.58	3.55	3.05
20	4.12	3.65	4.00	4.93	5.20	5.58	5.27	3.81
25	4.85	4.00	6.00	7.50	7.80	7.93	6.90	4.70
30	-	6.80	7.00	8.00	8.00	-	-	-

Least squares is used to obtain the values of $C_{D_0}(M)$ and $k(M)$ at each point. Then polynomial regression curve fitting is applied to derive the expressions for the functions $C_{D_0}(M)$, $k(M)$ and their derivatives $\partial C_{D_0}/\partial M$, $\partial k/\partial M$.

By the end of each boosting phase, the velocity is usually in the hypersonic range. Since theoretically C_{D_0} and k tend asymptotically to constant values when $M \to \infty$, these functions are set constant for $M \geqq 6.0$.

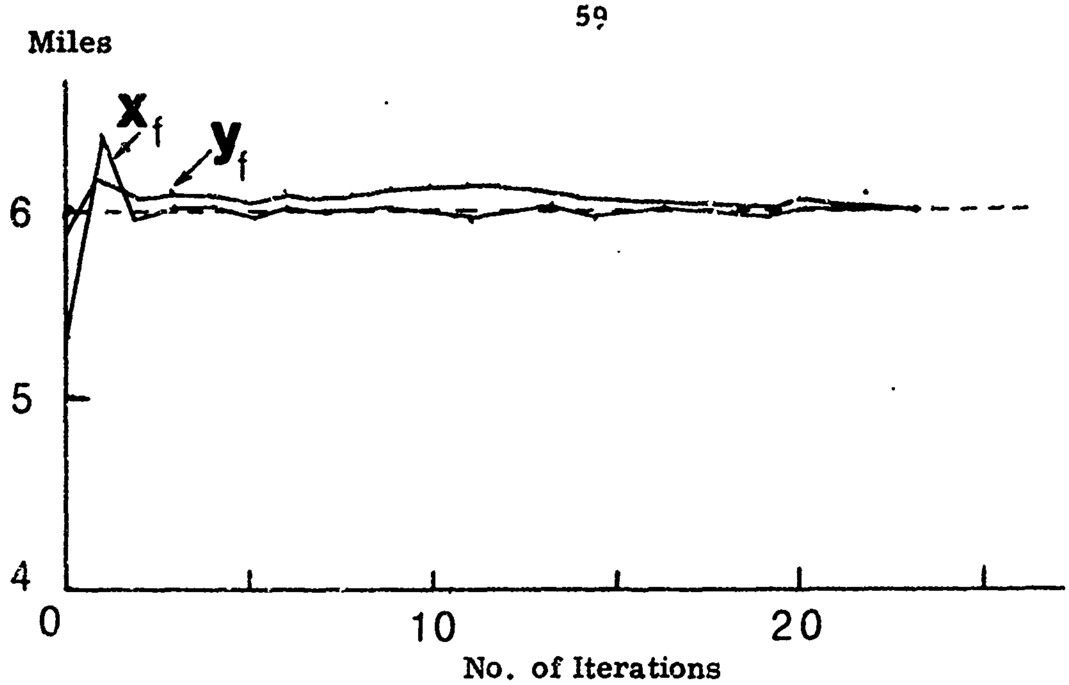


Fig. IV. 1 Convergence of the Final Coordinates (Problem 4)

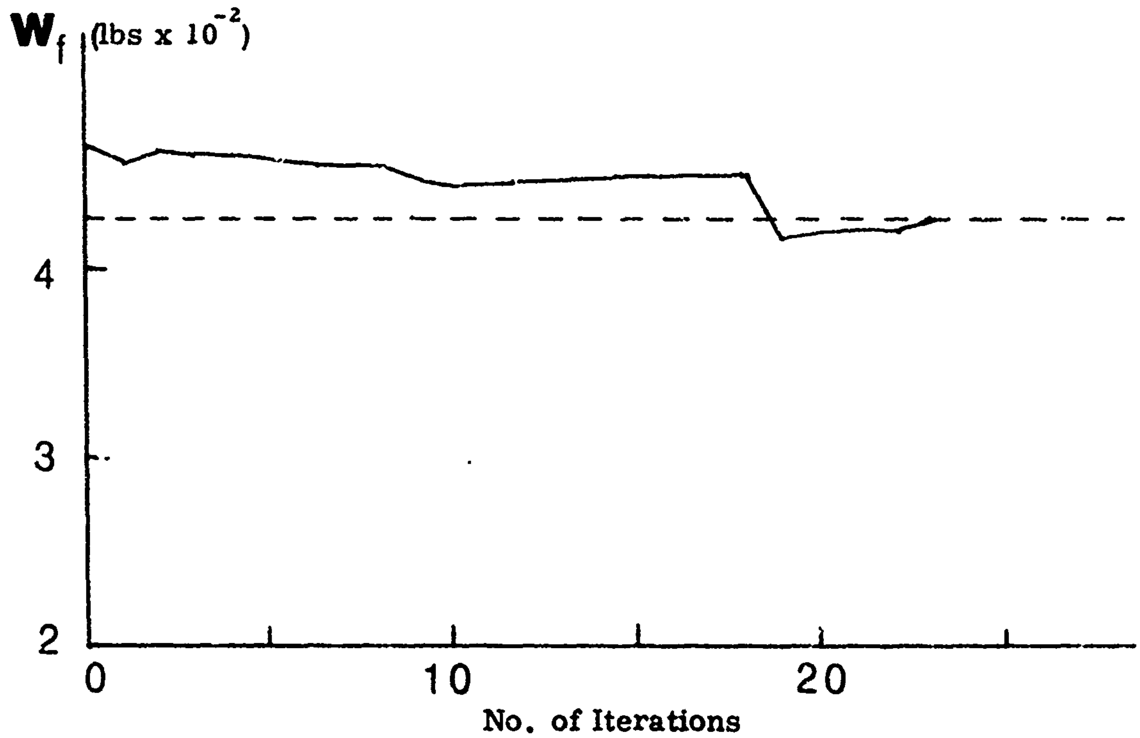


Fig. IV. 2 Convergence of the Final Weight (Problem 4)

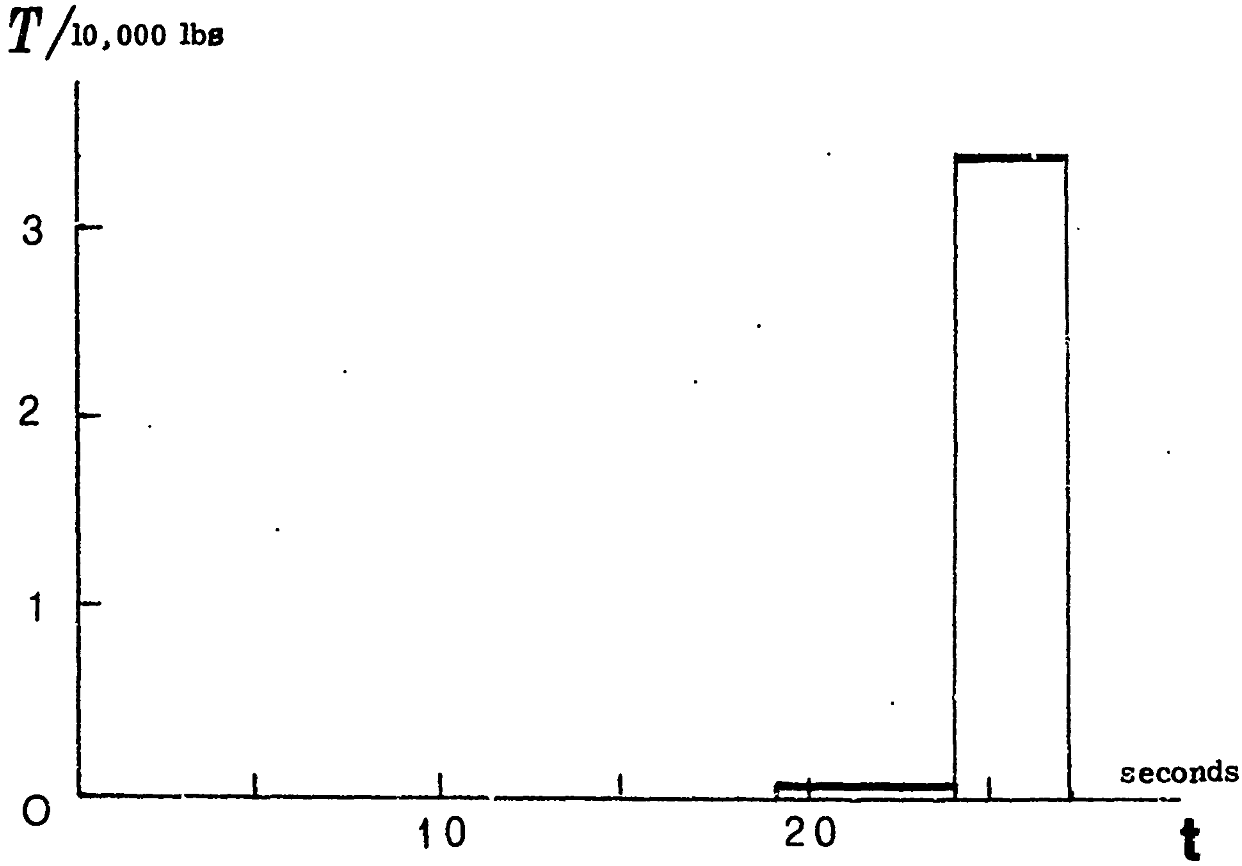


Fig.IV.3 Optimal Thrust Profile (Problem 4)

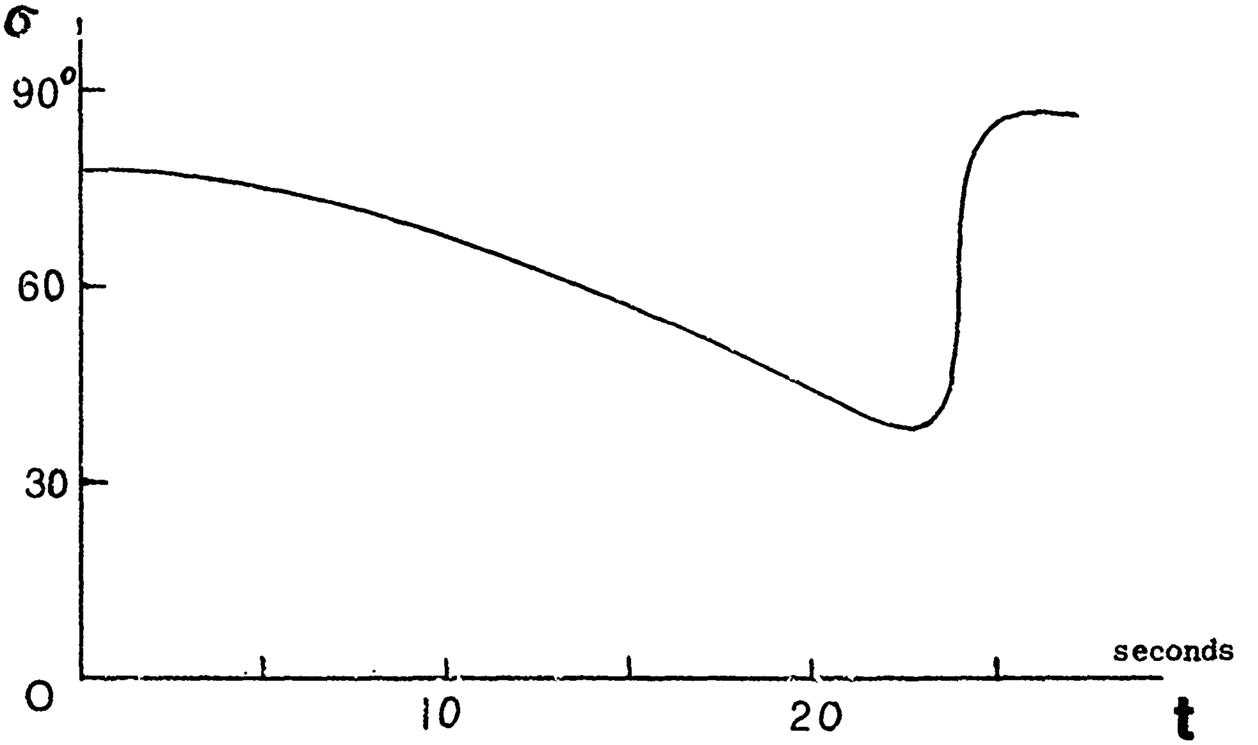


Fig.IV.4 Optimal Bank Angle (Problem 4)

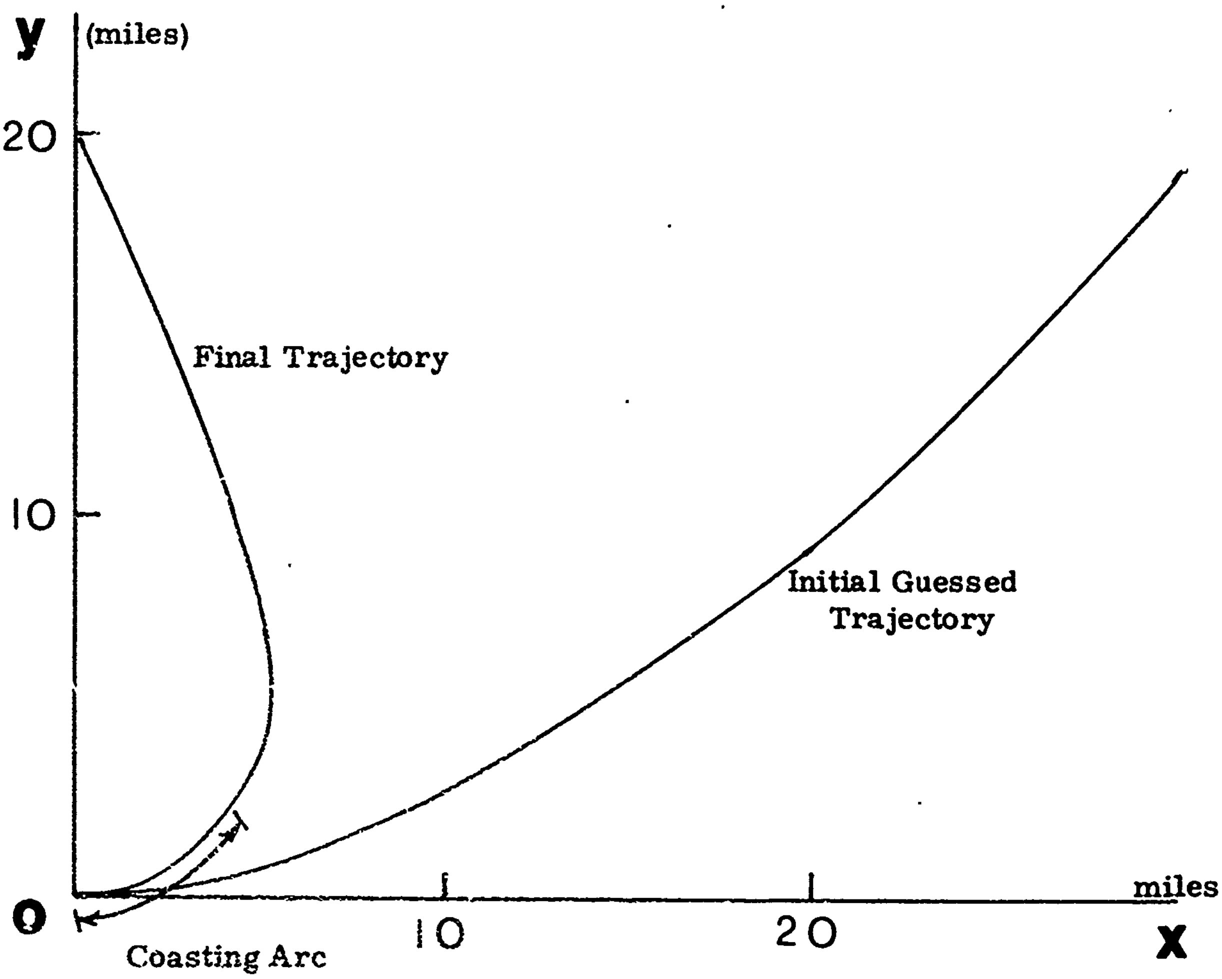


Fig. IV. 5 Optimal Trajectory with Initial Guessed Trajectory (Problem 5)

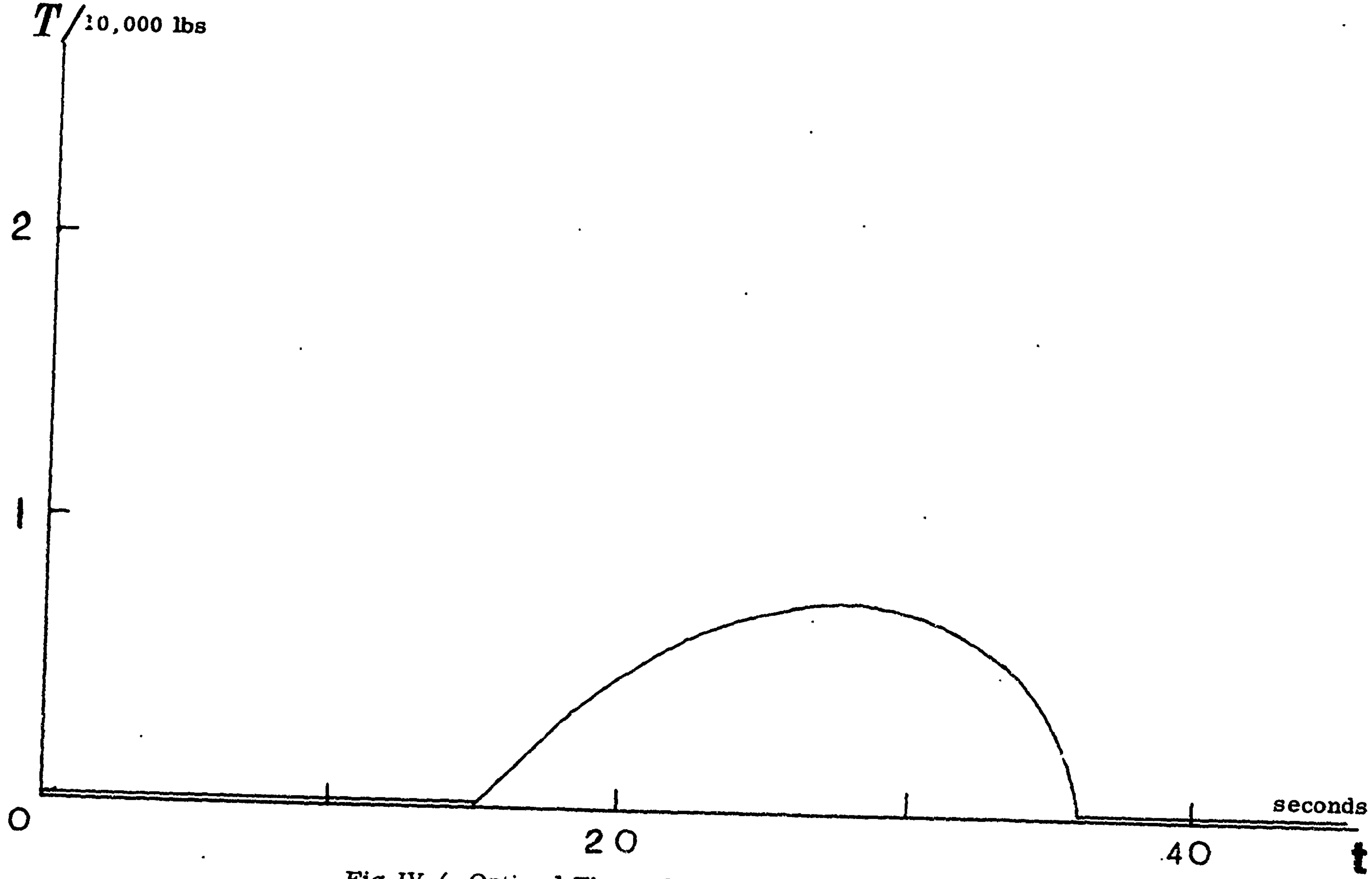


Fig. IV. 6 Optimal Thrust Profile (Problem 5)

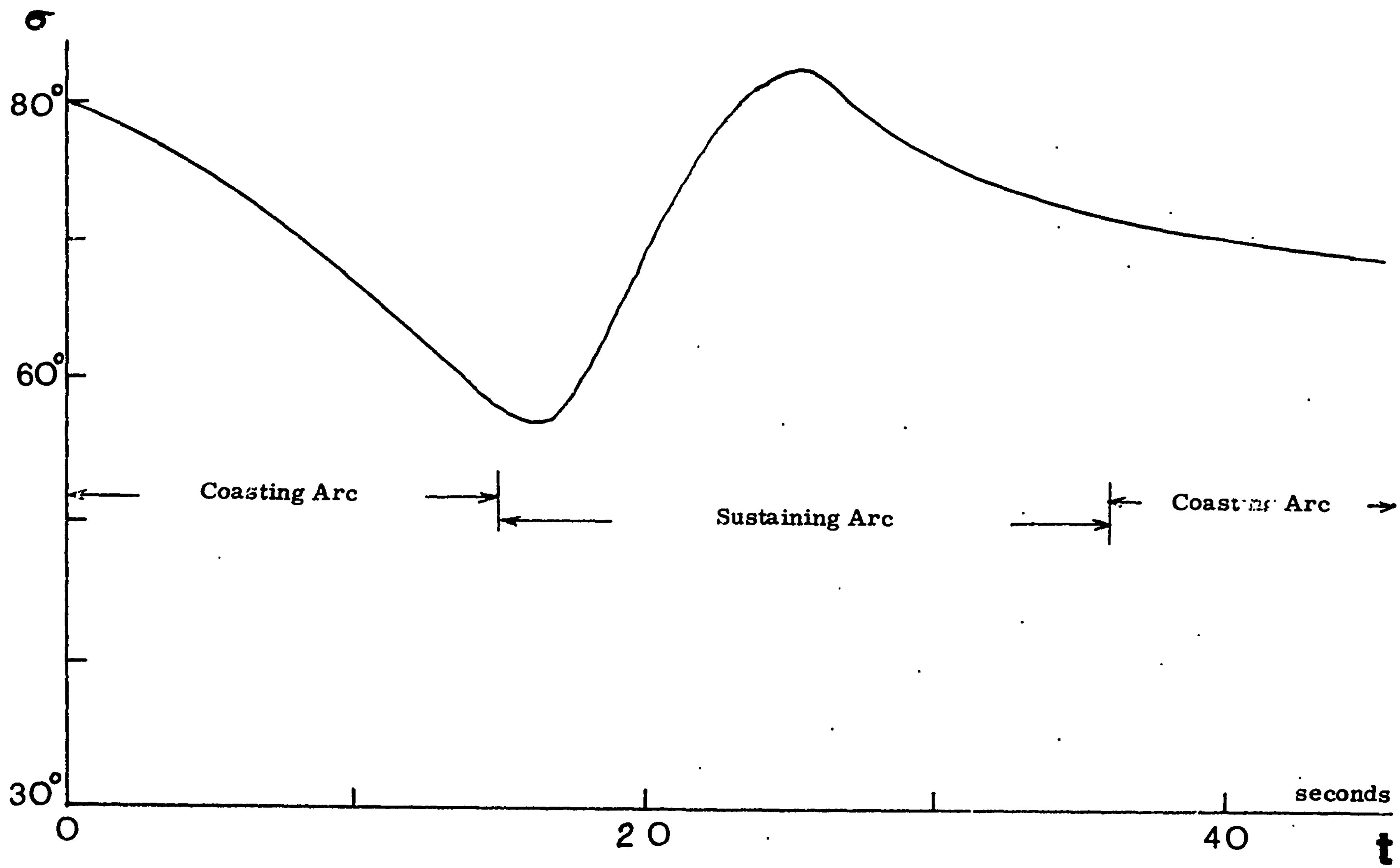


Fig. IV.7 Optimal Bank Angle (Problem 5)

V. QUALITATIVE ANALYSIS AND CONCLUSIONS

One of the objectives of this study is to determine the conditions under which a variable thrust magnitude program is optimal, and in that case, its relationship with missile parameters and trajectory. In the general case, such a relationship was displayed by Eq. (II.32) for the rocket motor with constant exhaust velocity. For horizontal, rectilinear flight, the variable thrust magnitude was given explicitly in terms of the flight velocity in Eq. (III.70), and for the case of turning flight with free terminal positions, it was obtained through the Eq. (III.85) for flight at maximum lift and the Eq. (III.109) for flight using variable lift coefficient. In all these cases, the thrust magnitude is always decreasing along the sustaining flight path and furthermore, it remains at low level for low Mach number. These conclusions have been substantiated by an independent numerical study carried out in Chapter IV.

If an ideal rocket motor where the thrust magnitude level is fully controllable (motor represented by the system (S_1) in Chapter II) is available, then the trajectory can be flown optimally. In practice, the rocket motor is preprogrammed and in this chapter we shall compare the performances of some preprogrammed motors with an ideal motor.

We can classify rocket motors used in Air Force missions, in order of increasing thrust controllability, as follows (Fig. V.1).

1. Preprogrammed motors. The thrust profile is fixed at the time of manufacture.

2. Preprogrammed pulse motors. The thrust profile is fixed at the time of manufacture; however, the off time between pulses may be controlled.

3. Stop-restart motors. This system permits motor shutdown and restart at any time on command. The thrust level is either rated thrust or zero.

4. Step-thrust motors. This system has the capability of providing thrust at more than two discrete levels. The on-off times and the order of

thrust levels are variable on command.

5. Continuously variable motor. This system, idealized by our model (S_1) offers the capability to provide any thrust time profile within specific limits of thrust and rate change of thrust.

As has been displayed explicitly by the equation (II.32), whenever variable thrust profile is optimal, it var... s function of the state variables $\vec{r}$, $\vec{V}$ and m and the adjoint components $\vec{p}_r$ and $\vec{p}_V$. The adjoint p_m is involved only in the switching function K (Eq. II.21) indicating the timing for stop and restart. It is known that the adjoint $\vec{p}_r$ and $\vec{p}_V$ depend on the terminal conditions. Hence for each specified mission we have a resulting optimal trajectory and a specified thrust profile.

The numerical examples given in Chapter IV clearly show that, unlike the simple case of horizontal rectilinear flight, the number of variables involved in the optimization problem for minimum time turning flight requires an important program for trajectory analysis in order to do an adequate comparative qualitative analysis among the different types of motors. Although this task is laborious, we believe that with our analytical results and numerical programs, the analysis can be easily carried out if sufficient computational time is allowed.

In this concluding chapter we shall give this comparative analysis for two specified problems given in Chapter IV, namely problem 2 and 4.

We have seen in problem 4 that, by constraining the final position (intercept problem), to meet the end-conditions, the trajectory usually includes a sustaining arc, where variable thrust is used. In general the variable thrust level is low compared to the maximum thrust. Hence, in the rockets used for the comparison we shall consider constant thrust at different intermediate levels with variable on-off time. The results of the comparative analysis are given in the tables below.

Case 1:

$X_0 = 0$, $Y_0 = 0$, $\beta_0 = 0$, $V_0 = 2136.2$ ft/sec, $W_0 = 861$ lbs

X_f = free, Y_f = free, $\beta_f = 135°$, $V_f \geqq 1000$ ft/sec, $W_f = 434$ lbs

Optimal Thrust Profile = Boost-Coast

Minimum time = 9.49 seconds

Comparison with Pre-programmed Motors

Motor	Thrust Profile (T in lbs)	Time (sec)
1	Boost-Coast $T = 34{,}000 \quad 0 \leqq t \leqq 3.14$ $T = 0 \quad t > 3.14$	9.49 (minimum)
2	Sustain-Coast $T = 12{,}500 \quad 0 \leqq t \leqq 9$ $T = 0 \quad t > 9$	11.2
3	Sustain-Coast $T = 6{,}640 \quad 0 \leqq t \leqq 6.25$ $T = 20{,}000 \quad 6.25 < t \leqq 9.25$ $T = 0 \quad t > 9.25$	12.04
4	Sustain-Boost-Coast $T = 7{,}900 \quad 0 \leqq t \leqq 8.25$ $T = 34{,}000 \quad 8.25 < t \leqq 9.25$ $T = 0 \quad t > 9.25$	12.17
5	Sustain-Boost-Coast $T = 3{,}230 \quad 0 \leqq t \leqq 7.25$ $T = 34{,}000 \quad 7.25 < t \leqq 9.25$ $T = 0 \quad t > 9.25$	13.37

The comparison in this case shows the influence of the timing and the order in the sequence of arcs.

Case 2:

$X_0 = 0$, $Y_0 = 0$, $\beta_0 = 0$, $V_0 = 2136.2$ ft/sec, $W_0 = 861$ lbs

$X_f = 31680$ ft, $Y_f = 31680$ ft, β_f = free, $V_f \geqq 1000$ ft/sec, $W_f = 434$ lbs

Optimal Thrust Profile = Coast-Sustain-Boost

Minimum Time = 27.16 seconds

Comparison with Pre-programmed Motors

Motor	Thrust Profile (T in lbs)	Time (sec)
1	Coast-Sustain-Boost (Fig. IV.3)	27.16 (minimum)
2	Coast-Boost $T = 0 \quad 0 \leqq t \leqq 24$ $T = 34{,}000 \quad t > 24$	27.20
3	Coast-Sustain-Boost $T = 0 \quad 0 \leqq t \leqq 23$ $T = 4{,}250 \quad 23 < t \leqq 25$ $T = 34{,}000 \quad 25 < t \leqq 28$	28.157

The comparison in this case shows that the variable thrust can be adequately approximated by either a null thrust or a constant low level thrust. The problem of minimum time turning in horizontal flight has been chosen for our numerical analysis but the computer program can be adapted for other types of performance indices such as maximum coverage (reachable sets). Furthermore, it can be used to solve optimization problems for flight in three-dimensional space.

Also the analytical results presented in this report have been extended to the case of optimal aerodynamic and thrusting maneuvers for three dimensional flight in a general gravitational force field (Refs. 9-11).

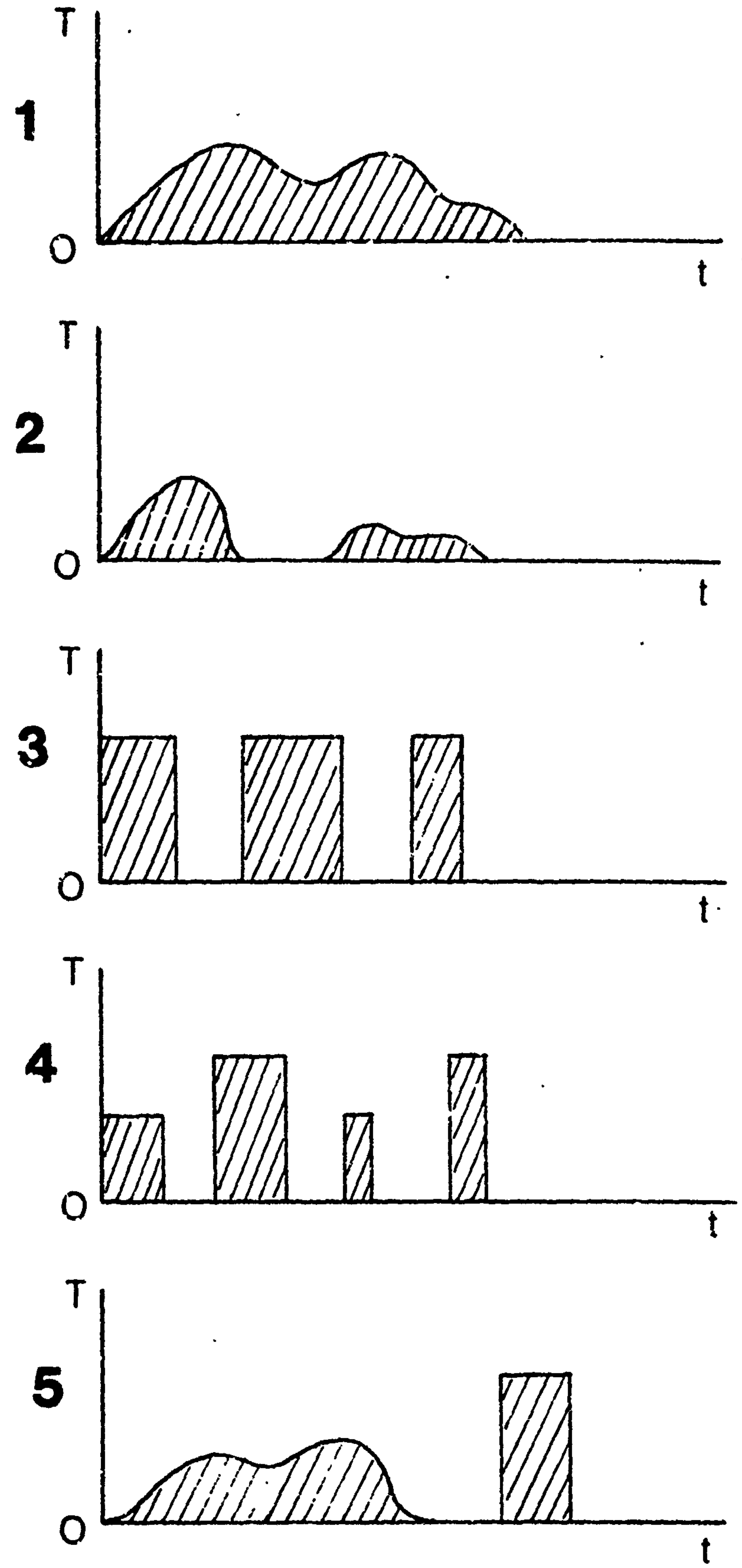


Fig. V. 1 Five Modes of Thrust Control in Rocket Technology

REFERENCES

1. Leitmann, G., "Variational Problems with Bounded Control Variables," Optimization Techniques, Ch. 5, Academic Press, 1962.
2. Marec, J.P., "Transferts Optimaux Entre Orbites Elliptiques Proches," D.Sc. Thesis, University of Paris, 1967.
3. Bryson, A.E., Jr., and Lele, M.M., "Minimum Fuel Lateral Turns at Constant Altitude," AIAA Journal, Vol.7, No.3, March 1969, pp. 559-560.
4. Salmon, D.M., Meier, L., and McReynolds, S.R., "A Preliminary Evaluation of Thrust Magnitude Control for Bomber Defense Missiles," Final Report, Systems Control, Inc., July 1971.
5. Hibbs, A., "Optimum Burning Program for Horizontal Flight," ARS. J., Vol. 22, No. 4, pp. 204-212, 1952.
6. Vinh, N.X., "Minimum Fuel Rocket Maneuvers in Horizontal Flight," Progress Report, AFOSR Grant No. 71-2129; May 1972; to appear in AIAA J.
7. Pagurek, B., and Woodside, C.M., "The Conjugate Gradient Method for Optimal Control Problems with Bounded Control Variables," Automatica, Vol. 4, pp. 337-349, 1968.
8. Lasdon, L.S., Mitter, S.K., and Waren, A.D., "The Conjugate Gradient Method for Optimal Control Problems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 2, pp. 132-138, 1967.
9. Vinh, N.X., "General Theory of Optimal Trajectory for Rocket Flight in a Resisting Medium," Progress Report, AFOSR Grant No. 71-2129. To appear in the Journal of Optimization Theory and Applications.
10. Vinh, N.X., "Integrals of the Motion for Optimal Trajectories in Atmospheric Flight," AIAA paper No. 72-931. Presented at the AIAA/AAS Astrodynamics Conference, Palo Alto, California, Sept. 11-12, 1972.
11. Griffin, J.W., Jr., and Vinh, N.X., "Optimal Three Dimensional Maneuvering of a Rocket Powered Hypervelocity Vehicle," IAF paper, presented at the 23rd Congress of the International Astronautical Federation, Vienna, Austria, Oct. 8-15, 1972.